Kinetic Physics Theory

Personal Notes

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Contents

1 Introduction

These notes cover my personal derivations and understanding of kinetic physics theory, which also serve as the theoretical foundation for my PhD project. They draw heavily on Peter Gary's book *Theory of Space Plasma Microinstabilities*, blended with my own takes and workings. So, let's go!!!!

2 The Basic Kinetic Equations

So, the general form of the kinetic equation is given by Bolzmann's equation, which describes the evolution of the distribution function $f(\mathbf{r}, \mathbf{v}, t)$ in phase space:

$$\frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \nabla f_j + \frac{q_j}{m_j}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_j = \left(\frac{\partial f_j}{\partial t}\right)_c, \tag{2.0.1}$$

where e_j is the charge of species j, m_j is the mass of species j, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and $\left(\frac{\partial f_j}{\partial t}\right)_c$ represents the collision term. Basically, this equation describes how the distribution function f_j changes over time due to the effects of particle motion, electromagnetic forces, and collisions.An And, the distribution function f_j has all the information that we need for kinetic physics!!!

With the distribution function, we can calculate fluid moments, which might be more familiar with everybody else, but they do not hold the physical essesce. Zeroth-order moment gives the number density n_j:

$$n_j \equiv \int f_j d^3v, \tag{2.0.2}$$

First-order moment gives the bulk velocity $\mathbf{U}_j$:

$$\mathbf{U}_j \equiv \frac{1}{n_j} \int \mathbf{v} f_j d^3v, \tag{2.0.3}$$

Second-order moment gives the pressure tensor $\mathbf{P}_j$:

$$\mathbf{P}_j \equiv m_j \int (\mathbf{v} - \mathbf{U}_j)(\mathbf{v} - \mathbf{U}_j) f_j d^3v, \tag{2.0.4}$$

and the scalar pressure p_j is given by:

$$p_j \equiv \frac{1}{3}\mathrm{Tr}(\mathbf{P}_j) = \frac{1}{3} m_j \int |\mathbf{v} - \mathbf{U}_j|^2 f_j d^3v. \tag{2.0.5}$$

Another import part of kinetic physics is the Maxwell's equations, which describe how the electric and magnetic fields evolve in space and time:

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{2.0.6}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2.0.7}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{2.0.8}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}, \tag{2.0.9}$$

where ρ is the charge density, c is the speed of light, and $\mathbf{J}$ is the current density. A physical way to remember Maxwell's equations:

1. **Gauss's law:** charges are the *sources* (and sinks) of electric fields. Think of field lines spreading out from a positive charge like water from a sprinkler — more charge, more flow through any enclosing surface.

2. **No magnetic monopoles:** magnetic field lines never start or end — they always form closed loops. Cut a magnet in half and you just get two smaller magnets, each with a north *and* south pole. There is no "magnetic charge".

3. **Faraday's law:** a changing magnetic field *creates* an electric field. The minus sign is Lenz's law in action — nature resists change: the induced electric field always opposes whatever is changing the magnetic flux.

4. **Ampère–Maxwell law:** there are *two* ways to make a magnetic field — an electric current (Ampère), *or* a changing electric field (Maxwell's crowning insight). The latter is what makes electromagnetic waves possible.

We always consider space plasma as collisionless, so the collision term is negligible, and we can write the Vlasov equation as:

$$\frac{\partial f_j}{\partial t} + \mathbf{v} \cdot \nabla f_j + \frac{q_j}{m_j}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \nabla_v f_j = 0. \tag{2.0.10}$$

So yeah, this equation is like the fundamental equation of kinetic physics in the solar wind. It is the "Lord of the Rings" of this field, and we are like little "Gollums" trying to understand it and its implications.

3 Electrostatic Waves in uniform Plasmas

We always try to start from the simplest case, which is electrostatic waves in uniform plasmas. Electrostatic means that the electric field is curl-free, i.e. $\nabla \times \mathbf{E} = 0$, which implies that the electric field can be expressed as the gradient of a scalar potential ϕ:

$$\mathbf{E} = -\nabla \phi. \tag{3.0.1}$$

Uniform plasmas means that the background plasma parameters (density, temperature, etc.) are constant in space and time.

3.1 Zeroth-order distributions: the Maxwellian distribution

$f_j^{(0)}$ of a Maxwellian can be written as:

$$f_j^{(0)}(\mathbf{v}) = f_j^{(M)}(v) = \frac{n_j}{(2\pi v_{th,j}^2)^{3/2}} \exp\left(-\frac{v^2}{2v_{th,j}^2}\right), \tag{3.1.1}$$

where $v_{th,j} \equiv \sqrt{k_B T_j / m_j}$ is the thermal speed of species j.

Of course the real plasma in space is much more complex than a simple Maxwellian, even more complex ones such as bi-Maxwellian, kappa are way too simplified. In fact, the author's PhD is basically about checking these complex distributions and their effects on the plasma instabilities, but we will start from the simplest case first.

3.2 Electrostatic waves in unmagnetized plasmas

The linear Vlasov equation in unmagnetized plasmas can be written as:

$$\frac{\partial f_j^{(1)}(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f_j^{(1)}(\mathbf{x},\mathbf{v},t)}{\partial \mathbf{x}} + \frac{e_j}{m_j}\mathbf{E}^{(1)}(\mathbf{x},t)\cdot\frac{\partial f_j^{(0)}(\mathbf{v})}{\partial \mathbf{v}} = 0. \tag{3.2.1}$$

Also, the electric field is given by Poisson's equation:

$$\nabla^2\phi^{(1)} = \nabla\cdot\mathbf{E}^{(1)}(\mathbf{x},t) = 4\pi\rho = 4\pi\sum_j e_j n_j^{(1)}(\mathbf{x},t), \tag{3.2.2}$$

Since the linear Vlasov equation has constant coefficients in a uniform plasma, we can Fourier-analyze the perturbations. The standard approach is a Fourier transform in space and a Laplace transform in time (to handle the initial-value problem properly). For a single Fourier mode, all first-order quantities vary as:

$$h^{(1)}(\mathbf{x},t) = h^{(1)}(\mathbf{k},\omega)\exp[i(\mathbf{k}\cdot\mathbf{x}-\omega t)], \tag{3.2.3}$$

where the components $\mathbf{k}$ is real but the frequency $\omega = \omega_r + i\gamma$ is complex, with the real part ω_r representing the oscillation frequency and the imaginary part γ representing the growth rate (if positive) or damping rate (if negative).

Then, the fluctuating electric field can be writen as:

$$\mathbf{E}^{(1)}(\mathbf{x},t) = -\nabla\phi^{(1)}(\mathbf{x},t), \tag{3.2.4}$$

then with some simple algebra and with Equation 3.2.1, we have:

$$f_j^{(1)}(\mathbf{k},\mathbf{v},\omega) = \frac{e_j}{m_j}\phi^{(1)}(\mathbf{k},\omega)\frac{\mathbf{k}}{\mathbf{k}\cdot\mathbf{v}-\omega}\cdot\frac{\partial f_j^{(0)}(\mathbf{v})}{\partial \mathbf{v}}. \tag{3.2.5}$$

With this, we can have the density perturbation $n_j^{(1)}$:

$$n_j^{(1)}(\mathbf{k},\omega) = \frac{e_j}{m_j}\phi^{(1)}(\mathbf{k},\omega)\int\frac{d^3v}{\mathbf{k}\cdot\mathbf{v}-\omega}\mathbf{k}\cdot\frac{\partial f_j^{(0)}(\mathbf{v})}{\partial \mathbf{v}}. \tag{3.2.6}$$

Here we define the susceptibility $K_j(\mathbf{k},\omega)$ as:

$$n_j^{(1)}(\mathbf{k},\omega) = -\frac{k^2\phi^{(1)}(\mathbf{k},\omega)}{4\pi e_j}K_j(\mathbf{k},\omega). \tag{3.2.7}$$

(P.S. Why define the susceptibility K_j? It is borrowed from dielectric theory: in a material you write $\mathbf{P} = \chi\mathbf{E}$, where χ encodes how the medium responds. A plasma is the same idea — K_j is the response function of species j, telling you how much density perturbation you get per unit potential. Summing over species gives the dielectric function $\varepsilon = 1 + \sum_j K_j$, and setting $\varepsilon = 0$ gives the dispersion relation. Landau (1946) just carried over a standard trick from condensed matter — treat the plasma like a dielectric, but the response involves velocity-space integrals instead of atomic dipoles.)

So that we have:

$$K_j(\mathbf{k}, \omega) = -\frac{\omega_j^2}{k^2 n_j} \int \frac{d^3 v}{\mathbf{k} \cdot \mathbf{v} - \omega} \mathbf{k} \cdot \frac{\partial f_j^{(0)}(\mathbf{v})}{\partial \mathbf{v}}; \tag{3.2.8}$$

or

$$K_j \mathbf{k}, \omega = -\frac{\omega_j^2}{n_j} \frac{\partial}{\partial \omega} \int \frac{d^3 v f_j^{(0)}(\mathbf{v})}{\mathbf{k} \cdot \mathbf{v} - \omega}. \tag{3.2.9}$$

Here, $\omega_j = \sqrt{4\pi n_j e_j^2 / m_j}$ is the plasma frequency of species j. Then, the dispersion relation can be written as:

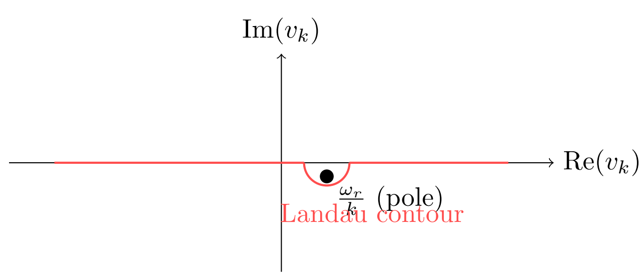
$$1 + \sum_j K_j(\mathbf{k}, \omega) = 0. \tag{3.2.10}$$

This is naturally derived from the Poisson's equation $(k^2 \phi^{(1)} = 4\pi \sum_j e_j (-k^2 \phi^{(1)}) K_j / (4\pi e_j))$. Expand Equation 3.2.10 with Equation 3.2.8, we have:

$$1 - \sum_j \frac{\omega_j^2}{n_j k^2} \int \frac{d v_k}{v_k - \omega / k_k} \frac{f_j^{(0)}(v_k)}{\partial v_k} = 0, \tag{3.2.11}$$

where k_k means that it could be positive or negative, where as k is always positive.

A really big issue happens here immediately: the integral has a singularity at $v_k = \omega / k_k$. This is the famous Landau singularity, which is the origin of Landau damping. So, why the singularity matters — and what is Landau damping? The integral has a pole at $v_k = \omega / k_k$, i.e. at the **phase velocity** of the wave. You cannot evaluate it as an ordinary Riemann integral. Landau's key move (1946) was to treat this as a Laplace-transform initial-value problem: ω is really complex, $\omega = \omega_r + i\gamma$. Since the pole is at $v_k = \omega / k_k$, its imaginary part is $\mathrm{Im}(v_k) = \gamma / k_k$. For a damped wave ($\gamma < 0$), the pole therefore sits *below* the real v_k axis in the complex v_k-plane. The integration contour must dip **underneath** the pole to properly continue the solution from the growing ($\gamma > 0$) side — this is the famous "Landau prescription". The detour around the pole picks up a residue, which is what gives the damping.



The contour choice determines the sign of the residue, which determines whether the wave damps or grows. The integral picks up an imaginary part — the dielectric function becomes complex, and $\varepsilon(\omega_r + i\gamma) = 0$ yields $\gamma \neq 0$. For a Maxwellian, $\partial f_0 / \partial v < 0$ at $v = \omega / k$, so $\gamma < 0$: the wave **damps**.

Physically: particles with $v \approx \omega/k$ are **resonant** — they surf the wave, like a surfer catching a perfect wave. Particles slightly *slower* than the phase velocity get pulled forward by the wave (gain energy $\rightarrow$ wave damps). Particles slightly *faster* get slowed down (lose energy $\rightarrow$ wave grows). For a Maxwellian there are more slower particles than faster ones, so the wave *loses* net energy — that's Landau damping. It is a collisionless process: energy transfers from the wave to particles via resonant wave-particle interaction, no collisions needed.

We spent quite a lot of text to illustrate the Landau damping, because it is one of the most important concepts in kinetic physics, and it is also a very counter-intuitive concept. Anyway, let's carry on. If we assume that the zeroth-order distribution is Maxwellian, then the susceptibility can be written in closed form:

$$K_j(\mathbf{k}, \omega) = -\frac{k_j^2}{2k^2} Z'(\zeta_j), \tag{3.2.12}$$

where $Z(\zeta)$ is the plasma dispersion function (Fried–Conte function), $\zeta_j \equiv \omega/(\sqrt{2}\, k\, v_{th,j})$. Physically, ζ_j tells you where the wave lands on the distribution: $\zeta_j \gg 1$ means the phase speed is far out on the tail (few resonant particles, weak damping), while $\zeta_j \sim 1$ means it samples the bulk (many resonant particles, strong damping). Expressing K_j in terms of $Z(\zeta)$ is powerful because the analytic properties of Z — its series expansions, asymptotic limits, and the imaginary part that encodes Landau damping — are all well tabulated.

P.S. The plasma dispersion function $Z(\zeta)$ (Fried & Conte, 1961)

Definition. For $\mathrm{Im}(\zeta) > 0$, with analytic continuation to $\mathrm{Im}(\zeta) \leq 0$ via the Landau contour:

$$Z(\zeta) \equiv \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{e^{-t^2}}{t - \zeta}\, dt.$$

Alternative forms:

$$\int_0^{\infty} e^{-\alpha^2 t^2 \pm i\beta t}\, dt = -\frac{i}{2|\alpha|}\, Z\!\left(\frac{\pm\beta}{2|\alpha|}\right), \qquad Z(\zeta) = 2i\, e^{-\zeta^2} \int_{-\infty}^{i\zeta} e^{-t^2}\, dt.$$

Key identities:

$$Z'(\zeta) = -2\big[1 + \zeta Z(\zeta)\big], \qquad Z(\zeta^*) = -Z^*(\zeta), \qquad Z(\zeta) + Z(-\zeta) = 2i\sqrt{\pi}\, e^{-\zeta^2}.$$

Expansions:

- **Small argument** ($|\zeta| \ll 1$, strong damping):

$$Z(\zeta) \approx i\sqrt{\pi} - 2\zeta + \tfrac{4}{3}\zeta^3 - \tfrac{8}{15}\zeta^5 + \cdots$$

- **Large argument** ($|\zeta| \gg 1$, weak damping):

$$Z(\zeta) \approx i\sqrt{\pi}\, \sigma\, e^{-\zeta^2} - \frac{1}{\zeta}\left(1 + \frac{1}{2\zeta^2} + \frac{3}{4\zeta^4} + \cdots\right),$$

  where $\sigma = 0$ for $\mathrm{Im}(\zeta) > 0$, $\sigma = 1$ for $\mathrm{Im}(\zeta) = 0$, and $\sigma = 2$ for $\mathrm{Im}(\zeta) < 0$. The σ parameter tracks which side of the pole you're on — same Landau contour logic as before.

3.2.1 Electron plasma waves

At frequencies above the electron plasma frequency, ions are too slow to respond, and only lighter species that contributs to the electron plasma waves. So, the conditions we make here are: $k \ll k_e$, $\zeta_e \gg 1$, and the electrons are nonresonant. Let's do a really detailed derivation here lol, in Peter Gary's book, it is quite simplified. The dispersion relation now only has electron contribution, so we have:

$$1 + K_e(\mathbf{k}, \omega) = 0, \tag{3.2.13}$$

which is equivalent to:

$$1 - \frac{k_e^2}{2k^2} Z'(\zeta_e) = 0. \tag{3.2.14}$$

At the limit of $k \ll k_e$ amd $\zeta_e \gg 1$, we can use the asymptotic expansion of $Z'(\zeta)$ (Just a Taylor expansion):

$$Z'(\zeta_e) \approx \frac{1}{\zeta_e^2} + \frac{3}{2\zeta_e^4} + ... - 2i\sqrt{\pi}\zeta_e \exp(-\zeta_e^2). \tag{3.2.15}$$

wich gives the dispersion relation:

$$D(k, \omega) = 1 - \frac{k_e^2}{2k^2} \left[\frac{1}{\zeta_e^2} + \frac{3}{2\zeta_e^4} + ... - 2i\sqrt{\pi}\zeta_e \exp(-\zeta_e^2) \right] = 0. \tag{3.2.16}$$

Now substitute $\zeta_e = \omega / (\sqrt{2} k v_{th,e})$ and take the real part, we have:

$$1 - \frac{k_e^2 v_{th,e}^2}{\omega^2} - \frac{3k_e^2 k^2 v_{th,e}^4}{\omega^4} = 0. \tag{3.2.17}$$

which is:

$$1 - \frac{\omega_e^2}{\omega^2} - \frac{3\omega_e^2 k^2 v_{th,e}^2}{\omega^4} = 0. \tag{3.2.18}$$

Under the long-wavelength limit $k \ll k_e$, thermal corrections are small, so $\omega^2 \approx \omega_e^2$, we can then have:

$$\omega_r^2 = \omega_e^2 + 3k^2 v_{th,e}^2. \tag{3.2.19}$$

This is the famous Bohm-Gross dispersion relation for electron plasma waves.

Then if we take the imaginary part of the dispersion relation, we have:

$$D_i(k, \omega) = \sqrt{\pi} \frac{k_e^2}{k^2} \zeta_e \exp\left(-\zeta_e^2\right) \tag{3.2.20}$$

Since $\omega = \omega_r + i\gamma$ and $|\gamma| \ll \omega_r$, we can Taylor expand it around ω_r:

$$D(k, \omega_r + i\gamma) \approx D_r(k, \omega_r) + i\gamma \frac{\partial D_r}{\partial \omega_r} + iD_i(k, \omega_r) = 0. \tag{3.2.21}$$

We know that $D_r(k, \omega_r) = 0$, so we have:

$$\gamma = -\frac{D_i(k, \omega_r)}{\partial D_r / \partial \omega_r|_{\omega_r}}. \tag{3.2.22}$$

The denominator, take $D_r \approx 1 - \frac{\omega_e^2}{\omega^2}$ (ignore higher-order thermal corrections), we have:

$$\frac{\partial D_r}{\partial \omega_r} = \frac{2\omega_e^2}{\omega_r^3} \approx \frac{2}{\omega_r}. \tag{3.2.23}$$

The numerator, put $\zeta_e = \omega/(\sqrt{2}kv_{th,e})$ and $k_e = \frac{\omega_e}{v_{th,e}}$, we have:

$$D_i = \frac{\pi}{2}\left(\frac{k_e}{k}\right)^3 \exp\left(-\frac{k_e^2}{2k^2}\right). \tag{3.2.24}$$

Then, γ can be written as:

$$\gamma = -\sqrt{\frac{\pi}{8}}\left(\frac{k_e}{k}\right)^3 \frac{\omega_r^2}{\omega_e} \exp\left(-\frac{k_e^2}{2k^2}\right). \tag{3.2.25}$$

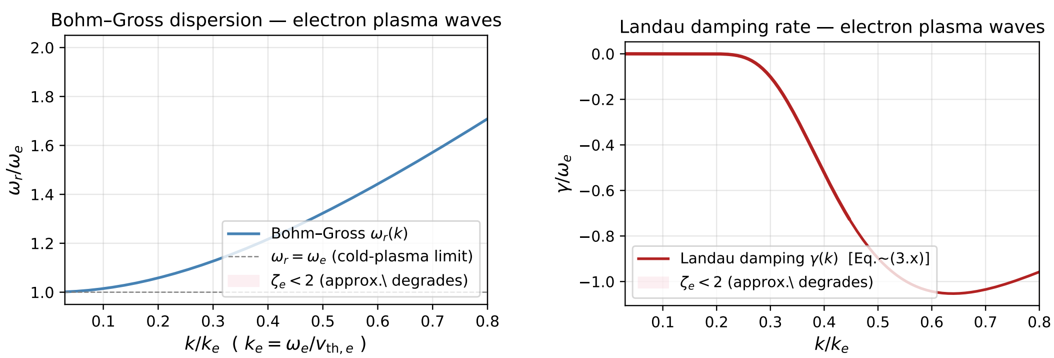


Figure 1: Left: Bohm–Gross dispersion relation $\omega_r^2 = \omega_e^2 + 3k^2 v_{\mathrm{th},e}^2$. Right: Landau damping rate $\gamma(k)$.

3.2.2 Ion Acoustic Waves

The ion acoustic wave is often considered a heavily damped wave under bi-Maxwellian assumptions. When $T_e \approx T_i$, it damps very quickly and is hard to observe in the solar wind. Only when $T_e \gg T_i$ can the ion acoustic wave be weakly damped and observable. The reason is simple: the ion-acoustic speed (which is also the phase speed of the wave) lies close to the ion thermal speed when the temperatures are comparable, so the wave is strongly damped by resonant ions. The $T_e \gg T_i$ assumption shifts the phase speed to be much larger than the ion thermal speed, opening a window for the wave to survive. However, the author of this note, which is also myself, haha, proved that in real measurement, the fine-scale structures of VDFs can allow the mode to exist even when $T_e \sim T_i$. Great work! But! We still quickly go through the derivation of the dispersion relations.

Both ions and electrons contribute to the ion acoustic wave, so we have:

$$1 + K_e(\mathbf{k}, \omega) + K_i(\mathbf{k}, \omega) = 0, \tag{3.2.26}$$

which is equivalent to:

$$1 - \frac{k_e^2}{2k^2} Z'(\zeta_e) - \frac{k_i^2}{2k^2} Z'(\zeta_i) = 0. \tag{3.2.27}$$

In this case, we have for electrons $|\zeta_e| \ll 1$, so:

$$Z'(\zeta_e) \approx -2 + 4\zeta_e^2 - i2\sqrt{\pi}\zeta_e. \tag{3.2.28}$$

For ions, we have $|\zeta_i| \gg 1$, so:

$$Z'(\zeta_i) \approx \frac{1}{\zeta_i^2} + \frac{3}{2\zeta_i^4} - 2i\sqrt{\pi}\zeta_i \exp(-\zeta_i^2). \tag{3.2.29}$$

Plugging the Z' expansions into the dispersion relation and keeping only the real parts (we'll come back to the imaginary part for the damping rate):

$$1 - \frac{k_e^2}{2k^2}\left(-2 + 4\zeta_e^2\right) - \frac{k_i^2}{2k^2}\left(\frac{1}{\zeta_i^2} + \frac{3}{2\zeta_i^4}\right) = 0. \tag{3.2.30}$$

Electron term. The leading piece is -2, giving k_e^2/k^2. The ζ_e^2 correction is $-(2k_e^2/k^2)\,\zeta_e^2 = -(k_e^2\omega_r^2)/(k^4 v_{\text{th},e}^2)$. For ion-acoustic waves the phase speed is far below the electron thermal speed, $\omega_r/k \ll v_{\text{th},e}$, so this correction is negligible — electrons respond quasi-statically, just tracking the instantaneous potential. Hence the electron contribution reduces to the simple shielding term k_e^2/k^2.

Ion term. Using $\zeta_i^{-2} = 2k^2 v_{\text{th},i}^2/\omega_r^2$ and $\zeta_i^{-4} = 4k^4 v_{\text{th},i}^4/\omega_r^4$,

$$-\frac{k_i^2}{2k^2}\left(\frac{2k^2 v_{\text{th},i}^2}{\omega_r^2} + \frac{6k^4 v_{\text{th},i}^4}{\omega_r^4}\right) = -\frac{\omega_i^2}{\omega_r^2} - \frac{3\omega_i^2 k^2 v_{\text{th},i}^2}{\omega_r^4}, \tag{3.2.31}$$

where we used $k_i^2 v_{\text{th},i}^2 = \omega_i^2$ and $k_i^2 v_{\text{th},i}^4 = \omega_i^2 v_{\text{th},i}^2$.

Assemble the real dispersion. Bringing electron and ion pieces together,

$$1 + \frac{k_e^2}{k^2} - \frac{\omega_i^2}{\omega_r^2} - \frac{3\omega_i^2 k^2 v_{\text{th},i}^2}{\omega_r^4} = 0. \tag{3.2.32}$$

This could also be expressed as:

$$\frac{k^2 + k_e^2}{k^2} - \frac{\omega_i^2}{\omega_r^2}\left(1 + \frac{3\omega_i^2 k^2 v_{th,i}^2}{\omega_r^2}\right) = 0. \tag{3.2.33}$$

Then we have:

$$\frac{\omega_r^2}{k^2} = \frac{\omega_i^2}{k_e^2(1 + k^2/k_e^2)}\left(1 + \frac{3k^2 v_{th,i}^2}{\omega_r^2}\right). \tag{3.2.34}$$

Note that actually the the item $3k^2 v_{th,i}^2/\omega_r^2$ is a small correction, so we can actually get $\omega_r \approx k^2\omega_i^2/k_e^2(1 + k^2/k_e^2)$, and put that back to the small correction item, we have:

P.S. I know, I know — mathematicians would kill themselves if they saw this. But this is *physics*, we don't care. This iterative "plug it back in" calculation is a bit ridiculous, I know, but it works.

$$\frac{3k^2 v_{th,i}^2}{\omega_r^2} \approx \frac{3k^2 v_{th,i}^2}{\frac{k^2\omega_i^2}{k_e^2(1+k^2/k_e^2)}} = 3v_{th,i}^2\frac{k_e^2}{\omega_i^2}(1 + k^2/k_e^2). \tag{3.2.35}$$

Put these back, we have the final dispersion relation in the standard form:

$$\frac{\omega_r^2}{k^2} = \frac{\omega_i^2}{k_e^2}\frac{1}{1+k^2/k_e^2} + 3v_{th,i}^2, \tag{3.2.36}$$

which is equivalent to:

$$\frac{\omega_r^2}{k^2} \approx \frac{3T_i}{m_i} + \frac{T_e}{m_i}\frac{1}{1+k^2/k_e^2}. \tag{3.2.37}$$

Damping rate (imaginary part).

Now we carry the imaginary pieces of Z' through the same machinery to obtain the Landau damping rate. Writing the full dispersion function as $D(k,\omega) = D_r(k,\omega)+iD_i(k,\omega)$ and assuming $|\gamma| \ll \omega_r$, we Taylor-expand around ω_r as before:

$$D(k,\omega_r + i\gamma) \approx D_r(k,\omega_r) + i\gamma\,\frac{\partial D_r}{\partial \omega_r} + iD_i(k,\omega_r) = 0, \tag{3.2.38}$$

which, with $D_r(k,\omega_r) = 0$, yields the standard weak-growth/damping formula

$$\gamma = -\frac{D_i(k,\omega_r)}{\partial D_r/\partial \omega_r}. \tag{3.2.39}$$

Imaginary part D_i. From the Z' expansions used for the real dispersion, we now keep the imaginary terms. For electrons ($|\zeta_e| \ll 1$):

$$Z'(\zeta_e) \approx -2 + 4\zeta_e^2 \;-\; i\,2\sqrt{\pi}\,\zeta_e, \tag{3.2.40}$$

so the electron contribution to D_i is

$$D_i^{(e)} = -\frac{k_e^2}{2k^2}\big(-2\sqrt{\pi}\,\zeta_e\big) = \sqrt{\pi}\,\frac{k_e^2}{k^2}\,\zeta_e. \tag{3.2.41}$$

Substituting $\zeta_e = \omega_r/(\sqrt{2}\,k\,v_{\mathrm{th},e})$ gives

$$D_i^{(e)} = \sqrt{\frac{\pi}{2}}\,\frac{k_e^2\,\omega_r}{k^3\,v_{\mathrm{th},e}} = \sqrt{\frac{\pi}{2}}\,\frac{\omega_e^2\,\omega_r}{k^3\,v_{\mathrm{th},e}^3}, \tag{3.2.42}$$

where we used $k_e^2 = \omega_e^2/v_{\mathrm{th},e}^2$.

For ions ($|\zeta_i| \gg 1$):

$$Z'(\zeta_i) \approx \frac{1}{\zeta_i^2} + \frac{3}{2\zeta_i^4} \;-\; i\,2\sqrt{\pi}\,\zeta_i\,e^{-\zeta_i^2}, \tag{3.2.43}$$

giving

$$D_i^{(i)} = -\frac{k_i^2}{2k^2}\Big(-2\sqrt{\pi}\,\zeta_i\,e^{-\zeta_i^2}\Big) = \sqrt{\pi}\,\frac{k_i^2}{k^2}\,\zeta_i\,e^{-\zeta_i^2}. \tag{3.2.44}$$

In the $T_e \gg T_i$ regime, $\zeta_i = c_s/(\sqrt{2}\,v_{\mathrm{th},i}) \sim \sqrt{T_e/(2T_i)} \gg 1$, so the factor $e^{-\zeta_i^2}$ suppresses the ion contribution exponentially. Electron Landau damping therefore dominates:

$$D_i \approx D_i^{(e)} = \sqrt{\frac{\pi}{2}}\,\frac{\omega_e^2\,\omega_r}{k^3\,v_{\mathrm{th},e}^3}. \tag{3.2.45}$$

Derivative $\partial D_r/\partial\omega_r$. From the real dispersion assembled earlier (keeping the leading electron and ion terms),

$$D_r \approx 1 + \frac{k_e^2}{k^2} - \frac{\omega_i^2}{\omega_r^2}, \tag{3.2.46}$$

so

$$\frac{\partial D_r}{\partial \omega_r} = \frac{2\omega_i^2}{\omega_r^3}. \tag{3.2.47}$$

Using the resonance condition $D_r = 0$, i.e. $\omega_i^2/\omega_r^2 = 1 + k_e^2/k^2$, we rewrite this as

$$\frac{\partial D_r}{\partial \omega_r} = \frac{2}{\omega_r}\Big(1 + \frac{k_e^2}{k^2}\Big). \tag{3.2.48}$$

For ion-acoustic waves we have $k^2 \ll k_e^2$ (long wavelength compared to the Debye length), so $1 + k_e^2/k^2 \approx k_e^2/k^2$, and

$$\frac{\partial D_r}{\partial \omega_r} \approx \frac{2k_e^2}{k^2\,\omega_r}. \tag{3.2.49}$$

Assembling γ. Substituting Equations (3.2.45) and (3.2.49) into (3.2.39):

$$\begin{aligned}
\gamma &= -\sqrt{\frac{\pi}{2}}\,\frac{\omega_e^2\,\omega_r}{k^3\,v_{\mathrm{th},e}^3} \Bigg/ \frac{2k_e^2}{k^2\,\omega_r} \\
&= -\sqrt{\frac{\pi}{2}}\,\frac{\omega_e^2\,\omega_r}{k^3\,v_{\mathrm{th},e}^3}\cdot\frac{k^2\,\omega_r}{2k_e^2} \\
&= -\sqrt{\frac{\pi}{8}}\,\frac{\omega_r^2}{k\,v_{\mathrm{th},e}},
\end{aligned} \tag{3.2.50}$$

where the $\omega_e^2/k_e^2 = v_{\mathrm{th},e}^2$ factors cancelled. Finally, using the ion-acoustic dispersion $\omega_r \approx kc_s$ with $c_s^2 = T_e/m_i$, we obtain the damping rate in its standard form:

$$\boxed{\gamma \approx -\sqrt{\frac{\pi}{8}}\,\frac{c_s}{v_{\mathrm{th},e}}\,\omega_r \;=\; -\sqrt{\frac{\pi}{8}}\,\sqrt{\frac{m_e}{m_i}}\,\omega_r \qquad (T_e \gg T_i).} \tag{3.2.51}$$

This is why the wave survives only when $T_e \gg T_i$ — the phase speed $c_s \sim \sqrt{T_e/m_i}$ is then well above the ion thermal speed (so ions are non-resonant), but still small enough compared to $v_{\mathrm{th},e}$ that only a tiny fraction of electrons are resonant, keeping the damping weak.

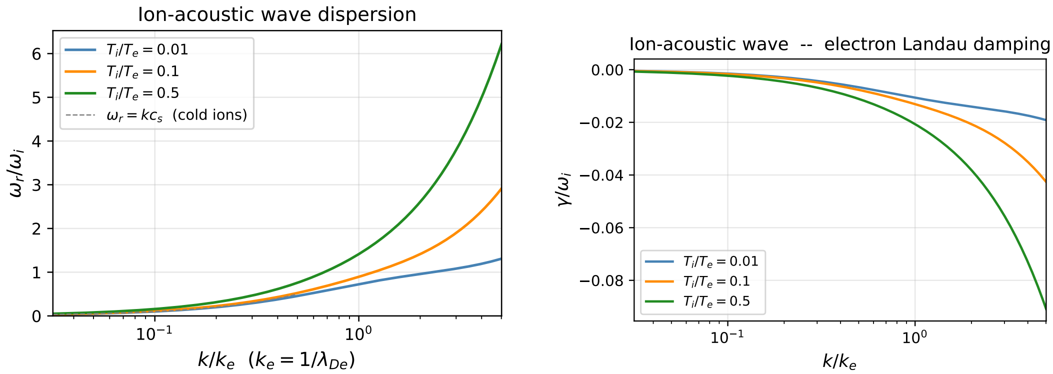


Figure 2: Left: ion-acoustic wave dispersion $\omega_r(k)$ for three values of T_i/T_e. Right: electron Landau damping rate $\gamma(k)$ (mass ratio $m_e/m_p = 1/1836$). At $T_i \sim T_e$ the phase speed drops toward the ion thermal speed and ion Landau damping (not shown) becomes the dominant loss channel, killing the wave.

3.3 Electrostatic waves in magnetized plasmas

Okay, now let's put magnetic field back into the picture. The linear Vlasov equation in a plasma with a uniform zeroth-order magnetc field $\mathbf{B}_0 = \hat{\mathbf{z}} B_0$:

$$\frac{\partial f_j^{(1)}(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\cdot\frac{\partial f_j^{(1)}}{\partial \mathbf{x}} + \frac{e_j}{m_j}\frac{\mathbf{v}\times\mathbf{B}_0}{c}\cdot\frac{\partial f_j^{(0)}(\mathbf{v})}{\partial \mathbf{v}} = -\frac{e_j}{m_j}\mathbf{E}^{(1)}(\mathbf{x},t)\cdot\frac{\partial f_j^{(0)}(\mathbf{v})}{\partial \mathbf{v}}. \quad (3.3.1)$$

We notice that, the left-hand side of the equation is the total derivative $df_j^{(1)}/dt$ along the unperturbed particle orbit, so it can be writen as:

$$\frac{d}{dt'} f_j^{(1)}(\mathbf{x}',\mathbf{v}',t') = -\frac{e_j}{m_j}\mathbf{E}^{(1)}(\mathbf{x}',t')\cdot\frac{\partial f_j^{(0)}(\mathbf{v}')}{\partial \mathbf{v}'}, \quad (3.3.2)$$

we also have $\mathbf{E}^{(1)}(\mathbf{x}',t') = -\nabla'\phi^{(1)}(\mathbf{x}',t')$, so:

$$f_j^{(1)}(\mathbf{x},\mathbf{v},t) = \frac{e_j}{m_j}\int_{-\infty}^{t} dt' \nabla'\phi^{(1)}(\mathbf{x}',t')\cdot\frac{\partial f_j^{(0)}(\mathbf{v}')}{\partial \mathbf{v}'}. \quad (3.3.3)$$

Here, the primed variables show the unperturbed particle orbit of a charged particle in this uniform magnetic field. Express the first-order quantities in Fourier space, we have:

$$f_j^{(1)}(\mathbf{k},\mathbf{v},\omega) = \frac{e_j}{m_j}\phi^{(1)}(\mathbf{k},\omega)\int_{-\infty}^{t} dt' i\mathbf{k}\cdot\frac{\partial f_j^{(0)}(\mathbf{v}')}{\partial \mathbf{v}'}\exp\left[i\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}) - i\omega(t'-t)\right]. \quad (3.3.4)$$

Take the zeroth-order moment, we have the density perturbation:

$$n_j^{(1)}(\mathbf{k},\omega) = \frac{e_j}{m_j}\phi^{(1)}(\mathbf{k},\omega)\int d^3v \int_{-\infty}^{t} dt' i\mathbf{k}\cdot\frac{\partial f_j^{(0)}(\mathbf{v}')}{\partial \mathbf{v}'}\exp\left\{i\left[\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x}) - \omega(t'-t)\right]\right\}. \quad (3.3.5)$$

Now, similar to the unmagnetized case, we have the susceptibility $K_j(\mathbf{k},\omega)$:

$$K_j(\mathbf{k},\omega) = -\frac{\omega_j^2}{n_j k^2}\int d^3v \int dt' i\mathbf{k}\cdot\frac{\partial f_j^{(0)}}{\partial \mathbf{v}'}\exp\left\{i\left[\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})-\omega(t'-t)\right]\right\}. \tag{3.3.6}$$

Now we want to actually evaluate this for a Maxwellian. The trick is to use the Maxwellian's key property:

$$\frac{\partial f_j^{(M)}(\mathbf{v}')}{\partial \mathbf{v}'} = -\frac{\mathbf{v}'}{v_{th,j}^2} f_j^{(M)}(v'). \tag{3.3.7}$$

This is simple: the derivative of a Gaussian gives you back the Gaussian times $-\mathbf{v}'/v_{th,j}^2$. Nothing magical, but extremely useful.

Plug this into the susceptibility expression (the equation just above), and the $i\mathbf{k}\cdot\partial f_j^{(M)}/\partial\mathbf{v}'$ part becomes:

$$i\mathbf{k}\cdot\frac{\partial f_j^{(M)}(\mathbf{v}')}{\partial \mathbf{v}'} = -\frac{i\mathbf{k}\cdot\mathbf{v}'}{v_{th,j}^2} f_j^{(M)}(v'). \tag{3.3.8}$$

Now, here's the central move. Look at the time derivative of the exponential phase factor:

$$\frac{\partial}{\partial t'}\exp\left\{i\left[\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})-\omega(t'-t)\right]\right\} = i(\mathbf{k}\cdot\mathbf{v}'-\omega)\exp\left\{i\left[\mathbf{k}\cdot(\mathbf{x}'-\mathbf{x})-\omega(t'-t)\right]\right\}, \tag{3.3.9}$$

which rearranges to:

$$i\mathbf{k}\cdot\mathbf{v}'\exp\left\{i[\cdots]\right\} = \left(\frac{\partial}{\partial t'}+i\omega\right)\exp\left\{i[\cdots]\right\}. \tag{3.3.10}$$

In terms of $\tau = t'-t$ (so $\partial/\partial t' = \partial/\partial\tau$, $dt' = d\tau$), this becomes:

$$i\mathbf{k}\cdot\mathbf{v}'\exp\left\{i[\cdots]\right\} = \left(\frac{\partial}{\partial \tau}+i\omega\right)\exp\left\{i[\cdots]\right\}. \tag{3.3.11}$$

So the τ-integral in K_j becomes:

$$\int_{-\infty}^{0} d\tau\, i\mathbf{k}\cdot\frac{\partial f_j^{(M)}(\mathbf{v}')}{\partial \mathbf{v}'}\,e^{i[\cdots]} = -\frac{1}{v_{th,j}^2}\int_{-\infty}^{0} d\tau\, f_j^{(M)}(v')\, i\mathbf{k}\cdot\mathbf{v}'\, e^{i[\cdots]}. \tag{3.3.12}$$

A crucial observation: $f_j^{(M)}(v')$ is a constant of the unperturbed motion. The magnetic field $\mathbf{B}_0$ does no work, so the speed $v' = |\mathbf{v}'|$ is constant along the gyro-orbit — only the *direction* of $\mathbf{v}'$ changes. Since the Maxwellian depends only on v'^2, we have $f_j^{(M)}(v') = f_j^{(M)}(v)$ (independent of τ), and it can be pulled outside the τ-integral.

Using Equation (3.3.10) to replace $i\mathbf{k}\cdot\mathbf{v}'$,

$$\int_{-\infty}^{0} d\tau\, i\mathbf{k}\cdot\frac{\partial f_j^{(M)}(\mathbf{v}')}{\partial \mathbf{v}'}\,e^{i[\cdots]} = -\frac{f_j^{(M)}(v)}{v_{th,j}^2}\int_{-\infty}^{0} d\tau\,\left(\frac{\partial}{\partial\tau}+i\omega\right)e^{i[\cdots]}. \tag{3.3.13}$$

The total τ-derivative gives a boundary term:

$$\int_{-\infty}^{0} d\tau\,\frac{\partial}{\partial\tau}e^{i[\cdots]} = e^{i[\cdots]}\Big|_{\tau=0} - e^{i[\cdots]}\Big|_{\tau\to-\infty}. \tag{3.3.14}$$

At $\tau = 0$, $\mathbf{x}' = \mathbf{x}$, $t' = t$, so the phase is zero, giving $e^0 = 1$. At $\tau \to -\infty$, we invoke the standard Laplace-transform prescription: $\mathrm{Im}(\omega) > 0$ for causal response, so $e^{-i\omega\tau} = e^{i\omega|\tau|}$ decays as $e^{-\mathrm{Im}(\omega)|\tau|} \to 0$, and this boundary vanishes. (This is the same $\mathrm{Im}(\omega) > 0$ logic from the Landau contour earlier — initial conditions at $t = -\infty$ are forgotten.)

Thus:

$$\int_{-\infty}^{0} d\tau \; i\mathbf{k} \cdot \frac{\partial f_j^{(M)}(\mathbf{v}')}{\partial \mathbf{v}'} \; e^{i[\cdots]} = -\frac{f_j^{(M)}(v)}{v_{th,j}^2} \left[1 + i\omega \int_{-\infty}^{0} d\tau \; e^{ib_j(\tau,\omega)} \right], \tag{3.3.15}$$

where we defined $b_j(\tau, \omega) \equiv \mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}) - \omega\tau$.

Now plug this back into the expression for K_j:

$$\begin{aligned} K_j^{(M)}(\mathbf{k}, \omega) &= -\frac{\omega_j^2}{n_j k^2} \int d^3v \left\{ -\frac{f_j^{(M)}(v)}{v_{th,j}^2} \left[1 + i\omega \int_{-\infty}^{0} d\tau \; e^{ib_j(\tau,\omega)} \right] \right\} \\ &= \frac{\omega_j^2}{n_j k^2 v_{th,j}^2} \underbrace{\int d^3v \; f_j^{(M)}(v)}_{=n_j} + \frac{\omega_j^2}{n_j k^2 v_{th,j}^2} \; i\omega \int d^3v \; f_j^{(M)}(v) \int_{-\infty}^{0} d\tau \; e^{ib_j(\tau,\omega)}. \end{aligned} \tag{3.3.16}$$

Using $k_j^2 = \omega_j^2 / v_{th,j}^2$, the first term simplifies, and we arrive at:

$$K_j^{(M)}(\mathbf{k}, \omega) = \frac{k_j^2}{k^2} \left\{ 1 + \frac{i\omega}{n_j} \int d^3v \, f_j^{(M)}(v) \int_{-\infty}^{0} d\tau \; e^{ib_j(\tau,\omega)} \right\}, \tag{3.3.17}$$

where $b_j(\tau, \omega) = \mathbf{k} \cdot (\mathbf{x}' - \mathbf{x}) - \omega\tau$, $\tau = t' - t$. Now we need the explicit unperturbed particle orbit in a uniform field $\mathbf{B}_0 = B_0 \hat{\mathbf{z}}$. A charged particle gyrates around the magnetic field line: its perpendicular velocity rotates at the cyclotron frequency $\Omega_j \equiv e_j B_0 / (m_j c)$, while its parallel velocity is constant. The equations of motion give:

$$v_x' = v_\perp \cos(\Omega_j \tau + \phi), \qquad x' - x = \frac{v_\perp}{\Omega_j} \big[\sin(\Omega_j \tau - \phi) + \sin\phi \big], \tag{3.3.18}$$

$$v_y' = -v_\perp \sin(\Omega_j \tau + \phi), \qquad y' - y = \frac{v_\perp}{\Omega_j} \big[\cos(\Omega_j \tau - \phi) - \cos\phi \big], \tag{3.3.19}$$

$$v_z' = v_z, \qquad z' - z = v_z \tau, \tag{3.3.20}$$

where $v_\perp = \sqrt{v_x^2 + v_y^2}$ is the perpendicular speed, ϕ is the initial gyrophase, and $\tau = t' - t$ (with $\tau \le 0$ since we integrate backward in time). Note that Ω_j carries the sign of the charge: electrons and ions gyrate in opposite senses. With generality, we can assume that $k_\perp = k_y$, so we have:

$$b_j(\tau, \omega) = \frac{k_y v_\perp}{\Omega_j} \big[\cos(\Omega_j \tau - \phi) - \cos\phi \big] + (k_z v_z - \omega)\tau. \tag{3.3.21}$$

The remaining velocity-space integrals — Jacobi–Anger expansion of the phase factor, integration over τ, ϕ, $v_\perp$, and the v_z Landau contour — are carried out in detail in Appendix A. Here we state the final result for the magnetized electrostatic susceptibility of a

Maxwellian plasma:

$$\boxed{K_j^{(M)}(\mathbf{k},\omega) = \frac{k_j^2}{k^2}\left\{1 + \frac{i\omega}{|k_z|v_j}\, e^{-\lambda_j} \sum_{m=-\infty}^{\infty} I_m(\lambda_j)\, Z(\zeta_j^m)\right\},} \tag{3.3.22}$$

with $\lambda_j \equiv k_\perp^2 v_{th,j}^2/\Omega_j^2 = k_\perp^2\rho_j^2$ (where $\rho_j = v_{th,j}/\Omega_j$ is the thermal gyroradius), $\zeta_j^m \equiv (\omega + m\Omega_j)/(\sqrt{2}\,|k_z|v_{th,j})$, $\Omega_j \equiv e_jB_0/(m_jc)$ (signed), and $v_j = \sqrt{2}\,v_{th,j}$. The dispersion relation for electrostatic waves in a magnetized plasma is then $1+\sum_j K_j^{(M)} = 0$, same form as always.

Physics of this result. The sum over m is a sum over **cyclotron harmonics**. Each term m corresponds to a resonance $\omega - k_zv_z = m\Omega_j$: a particle with parallel velocity v_z sees the wave Doppler-shifted to its own frame, and if that shifted frequency matches an integer multiple of its gyrofrequency, the particle is in **cyclotron resonance** with the wave. The $I_m(\lambda_j)$ factor (a modified Bessel function) weights each harmonic by how strongly a particle with gyroradius ρ_j "samples" the perpendicular wavelength — when $k_\perp\rho_j \ll 1$, only $m = 0$ survives, recovering the unmagnetized result; when $k_\perp\rho_j \sim 1$, many harmonics contribute. The $Z(\zeta_j^m)$ encodes Landau damping on each harmonic: the resonant velocity is $v_z = (\omega + m\Omega_j)/k_z$, and the slope of the distribution at that velocity determines whether the wave damps or grows.

P.S. We are still just at a simplified version (uniform magnetic field) of a simplified version (Maxwellian) of a simplified version (linearized Vlasov) of the reality. And even this took us pages of Bessel functions and Landau contours to get through.

Speaking of heroic levels of difficulty: the new Fields medalist Yu Deng (2026) actually *rigorously derived* the Boltzmann equation — the collisional cousin of our Vlasov equation — straight from Newton's equations of motion for hard-sphere particles. That is: start with N billiard balls obeying $\mathbf{F} = m\mathbf{a}$, take the limit $N \to \infty$, and *prove* the kinetic equation emerges, with no hand-waving. This was Hilbert's 6th problem, open for over a century. Different equation (Boltzmann has collisions; ours is collisionless), but the same spirit — connecting the microscopic to the statistical. What a hero. Congrats to him!

3.3.1 Oblique propagation: ion acoustic and ion cyclotron modes

So far we have been looking at waves that propagate purely parallel ($k_\perp = 0$) or purely perpendicular ($k_z = 0$) to the magnetic field. Real plasmas are rarely so cooperative — wave vectors can point in any direction, and the angle θ between $\mathbf{k}$ and $\mathbf{B}_0$ fundamentally changes which modes exist and how they behave.

The geometry. When $\mathbf{k}$ is oblique, its perpendicular component $k_\perp = k\sin\theta$ samples the ion gyro-orbit. Figure 3 shows the setup: the wave travels at angle θ to $\mathbf{B}_0$, and an ion gyrating around the field line "sees" different wave phases at different points along its circular orbit. The degree to which the wave "feels" the gyromotion is controlled by the parameter $\lambda_i \equiv k_\perp^2\rho_i^2$, where $\rho_i = v_{\mathrm{th},i}/\Omega_i$ is the ion thermal gyroradius.

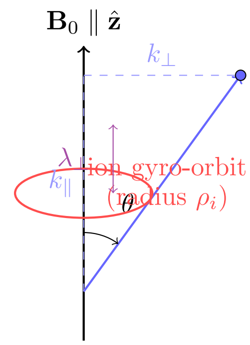


Figure 3: Oblique propagation geometry. The wave vector $\mathbf{k}$ makes an angle θ with the background magnetic field $\mathbf{B}_0$. The ion gyro-orbit (red ellipse) has radius $\rho_i = v_{\text{th},i}/\Omega_i$. When $k_\perp \rho_i \sim 1$, the ion samples different wave phases around its orbit — this is what turns on the cyclotron harmonic resonances.

How obliqueness turns on cyclotron harmonics. Recall our magnetized susceptibility $K_j^{(M)}$ from Equation (3.3.22): the ion response contains the sum $\sum_{m=-\infty}^{\infty} \Gamma_m(\lambda_i)\, Z(\zeta_i^m)$, where $\Gamma_m(\lambda) = e^{-\lambda} I_m(\lambda)$. Figure 4 shows these coupling coefficients.

When $\lambda_i \ll 1$ (i.e. $k_\perp \rho_i \ll 1$, nearly parallel propagation), only $\Gamma_0(0) = 1$ contributes — we recover the unmagnetized acoustic response. As λ_i increases toward ~ 1, the $m = \pm 1$ (fundamental cyclotron) channel opens. At larger λ_i, higher harmonics $m = \pm 2, \pm 3, \ldots$ sequentially turn on.

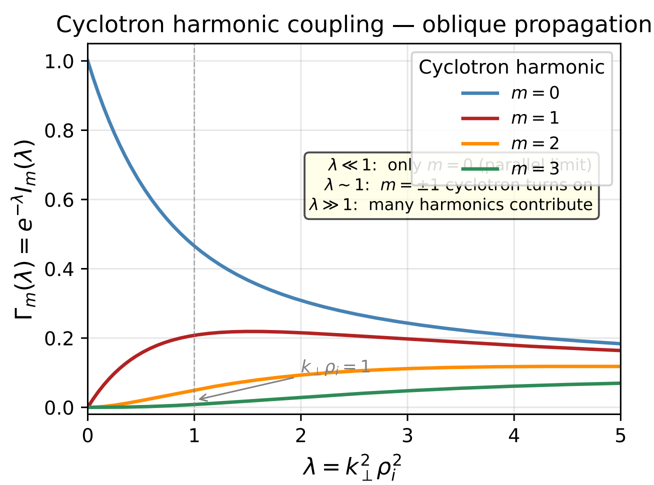


Figure 4: Cyclotron harmonic coupling coefficients $\Gamma_m(\lambda) = e^{-\lambda} I_m(\lambda)$. At $\lambda \ll 1$ (parallel limit) only $m = 0$ (acoustic/Landau channel) matters. At $\lambda \sim 1$ (oblique, $k_\perp \rho_i \sim 1$) the $m = \pm 1$ cyclotron resonance turns on. At $\lambda \gg 1$, many harmonics contribute.

Why is the ion cyclotron wave "usually" parallel? Two answers, one physical and one practical.

Physical: the fundamental cyclotron resonance $\omega - k_\parallel v_\parallel = \pm \Omega_i$ is cleanest at $k_\perp = 0$. With a perpendicular component, particles at different gyroradii see different wave phases along their orbit, which broadens the resonance and weakens the coupling. The strongest wave-particle energy exchange happens when the perpendicular wavelength is much larger than the ion gyroradius ($k_\perp \rho_i \ll 1$), so the wave electric field rotates in phase with every ion's gyromotion regardless of its Larmor radius. This condition is most easily satisfied at parallel propagation.

Practical: in the solar wind, ion cyclotron waves are most commonly *observed* propagating nearly parallel to $\mathbf{B}_0$, driven by the proton temperature anisotropy instability ($T_\perp > T_\parallel$), which has maximum growth at $\theta \approx 0°$. Textbooks therefore focus on the parallel case.

But oblique propagation is real and important. At exactly 90°, the electrostatic ion cyclotron wave (ion Bernstein mode) propagates purely perpendicularly with no Landau damping (since $k_\parallel = 0$, no particle can satisfy the Cherenkov resonance $\omega = k_\parallel v_\parallel$). At intermediate angles $0° < \theta < 90°$, the acoustic and cyclotron branches hybridize — the wave has both electrostatic and (in the full electromagnetic treatment) electromagnetic components, and the physics becomes richer than either limiting case alone.

Ion Bernstein modes. At exactly perpendicular propagation ($k_\parallel = 0$), the electrostatic ion cyclotron wave becomes what is known as the **ion Bernstein mode**. These are purely kinetic modes that exist in frequency bands near the harmonics of the ion cyclotron frequency, $\omega \approx n\Omega_i$ ($n = 1, 2, 3, \dots$). Physically, they are a finite-Larmor-radius (FLR) effect. In a strictly cold plasma, higher cyclotron harmonics ($n \geq 2$) do not even appear in the dispersion relation. It is only when ions possess thermal motion ($k_\perp \rho_i \sim 1$) that their finite orbits sample the spatial structure of the wave's electric field. This introduces Bessel functions into the dielectric response, mathematically opening up resonant poles at every harmonic $n\Omega_i$. The dispersion curves bend away from these exact harmonics depending on $k_\perp$, creating propagating bands separated by *stop bands* where the wave becomes evanescent. Because $k_\parallel = 0$, no particle can satisfy the Cherenkov resonance condition $\omega = k_\parallel v_\parallel$. Therefore, **there is no Landau or cyclotron damping** — Bernstein modes are undamped in a linear, collisionless plasma. This makes them observationally distinct: they are narrowband, long-lived, and appear in spacecraft spectrograms as extremely bright, discrete harmonic stripes (often related to equatorial noise in the magnetosphere). In practice, Bernstein modes can be driven unstable by temperature anisotropy, loss-cone, or ring distributions in the ion VDF, making them a powerful diagnostic of ion-scale kinetic physics in both space and laboratory plasmas.

The take-home: the propagation angle θ is a knob that controls which terms in the $\sum_m$ cyclotron-harmonic sum contribute. Parallel ($\theta \approx 0°$) isolates the $m = 0$ acoustic channel; perpendicular ($\theta \approx 90°$) isolates the $m = \pm 1$ cyclotron channel; and oblique accesses everything in between.

P.S. A note on electrostatic vs. electromagnetic.

Everything we have derived so far — the susceptibility $K_j^{(M)}$, the dispersion relation $1 + \sum_j K_j^{(M)} = 0$, and the cyclotron harmonic sum — is for **electrostatic** waves, where the electric field comes from a scalar potential ($\mathbf{E} = -\nabla\phi$) and Poisson's equation closes the system. This is the simplest entry point: one scalar equation, one unknown (ϕ), and the physics of Landau damping and cyclotron resonance is already fully on display.

In the **electromagnetic** case, the wave carries both $\mathbf{E}$ and $\mathbf{B}$ fluctuations, and you need the full set of Maxwell's equations (including $\nabla \times \mathbf{B}$ with the displacement current). The dispersion relation becomes a tensor equation $\mathbf{D}(\mathbf{k}, \omega) \cdot \delta\mathbf{E} = 0$, where $\mathbf{D}$ is a 3×3 matrix whose entries involve sums over Bessel functions — far messier, but necessary to capture the wave's polarization. The key physical difference is which cyclotron harmonic couples to which wave polarization. For **parallel** propagation in the EM case, the wave decoupling yields circularly polarized modes with intrinsic handedness. The **left-hand** circularly polarized wave (the Alfvén / EMIC branch) resonates strongly with ions via the fundamental $m = \pm 1$ harmonic (the sign depends on your convention for Ω_i). This differs from the electrostatic picture where both signs of m appear symmetrically in the sum because an electrostatic wave is strictly longitudinally polarized ($\mathbf{E} \parallel \mathbf{k}$) and carries no helicity. Oblique propagation in the full EM theory hybridizes the electrostatic and electromagnetic branches, which is what makes the solar wind's fluctuation spectrum so rich — but that is a story for another chapter.

4 Electrostatic component/ component instabilities in uniform plasmas

Instabilities occur because the plasma VDFs carry free energy: an isotropy, a drift, a bump, or some other non-Maxwellian feature that can be tapped by a wave.

4.1 Zeroth-order distributions: free energy source

The book, honestly, didn't say anything all that difficult. The key point to emphasize is that traditional theory (at least as presented in Peter Gary's book) is built on free-energy sources that can be expressed in closed mathematical form. That is, they use bi-Maxwellian-type distributions and evaluate the contributions from temperature anisotropies, drifts, secondary components, and so on. But reality is far more complex—especially in the solar wind, where we routinely observe fine-scale structures in the VDFs. These structures, particularly those near the resonance speeds, can have a dramatic effect on wave stability even if they look small. So, while traditional theory is a good starting point, it is not enough to explain what we actually measure. A good example, haha, is our paper published in 2026.

4.2 Electrostatic component/ component instabilities in unmagnetized plasmas

The basic theory is similar, let's go through it quickly here. The dispersion relation is always:

$$1 + \sum_j K_j(\mathbf{k}, \omega) = 0, \tag{4.2.1}$$

where the j-th species susceptibility is (Equation 3.2.9):

$$K_j(\mathbf{k}, \omega) = -\frac{\omega_j^2}{n_j} \frac{\partial}{\partial \omega} \int \frac{d^3 v f_j^{(0)}(\mathbf{v})}{\mathbf{k} \cdot \mathbf{v} - \omega}. \tag{4.2.2}$$

Throughout this section, the zeroth-order distribution is defined as drifting Maxwellians. We can express the susceptibility in terms of the plasma dispersion function $Z(\zeta)$:

$$K_j(\mathbf{k},\omega) = \frac{k_j^2}{k^2}\left[1+\zeta_j Z(\zeta_j)\right], \tag{4.2.3}$$

but here the argument of the plasma dispersion function is $\zeta_j = (\omega - \mathbf{k}\cdot\mathbf{v}_{oj})/(kv_{th,j})$, where $\mathbf{v}_{oj}$ is the drift velocity of the j-th species.

4.2.1 Electron / ion instabilities

In this scenario, the electron modes do not really change much, but the ion-acoustic mode can be driven unstable when the electron drift v_{oe} is sufficiently large. Here we derive the growth rate in detail.

The dispersion relation with drifting electrons. We start from the susceptibility for a drifting Maxwellian (Equation 3.2.9):

$$K_j(\mathbf{k},\omega) = \frac{k_j^2}{k^2}\big[1+\zeta_j Z(\zeta_j)\big], \tag{4.2.4}$$

with $\zeta_j = (\omega - \mathbf{k}\cdot\mathbf{v}_{oj})/(\sqrt{2}\,k\,v_{\mathrm{th},j})$. For electrons, $\mathbf{v}_{oe} \neq 0$; for ions we take $\mathbf{v}_{oi} = 0$ (ions stationary in the lab frame). The dispersion relation is $1 + K_e + K_i = 0$, which we split into real and imaginary parts as before:

$$\gamma = -\frac{D_i(k,\omega_r)}{\partial D_r/\partial\omega_r}, \qquad D \equiv 1 + K_e + K_i. \tag{4.2.5}$$

Real part — recovering the ion-acoustic dispersion. Under $T_e \gg T_i$ and $v_{oe} \ll v_{\mathrm{th},e}$, we have $|\zeta_e| \ll 1$ (the phase speed in the electron frame is far below the electron thermal speed). Using the small-argument expansion $Z(\zeta_e) \approx i\sqrt{\pi} - 2\zeta_e + \cdots$, we get

$$1 + \zeta_e Z(\zeta_e) \approx 1 - 2\zeta_e^2 = 1 - \frac{(\omega - \mathbf{k}\cdot\mathbf{v}_{oe})^2}{k^2 v_{\mathrm{th},e}^2} \approx 1, \tag{4.2.6}$$

where the ζ_e^2 correction is negligible for ion-acoustic phase speeds. For ions, $|\zeta_i| \gg 1$ (large argument): $1 + \zeta_i Z(\zeta_i) \approx -k^2 v_{\mathrm{th},i}^2/\omega^2$, giving $K_i \approx -\omega_i^2/\omega_r^2$. Thus the real dispersion is unchanged from the undrifted case:

$$D_r \approx 1 + \frac{k_e^2}{k^2} - \frac{\omega_i^2}{\omega_r^2} = 0, \qquad \omega_r \approx kc_s \ \ (k^2 \ll k_e^2), \tag{4.2.7}$$

and

$$\frac{\partial D_r}{\partial \omega_r} = \frac{2\omega_i^2}{\omega_r^3} = \frac{2}{\omega_r}\left(1 + \frac{k_e^2}{k^2}\right) \approx \frac{2k_e^2}{k^2\,\omega_r}. \tag{4.2.8}$$

Imaginary part — where the exponential comes from. Here is the key point the reader may have stumbled on. The *exact* imaginary part of the plasma dispersion function for real argument is

$$\mathrm{Im}[Z(x)] = \sqrt{\pi}\,e^{-x^2} \qquad (x \in \mathbb{R}), \tag{4.2.9}$$

which holds for *all* x, both small and large. Therefore, without any expansion at all,

$$\mathrm{Im}\big[\,\zeta_e Z(\zeta_e)\,\big] = \sqrt{\pi}\,\zeta_e\, e^{-\zeta_e^2}. \tag{4.2.10}$$

If one now blindly applies the small-argument expansion $e^{-\zeta_e^2} \approx 1$, one gets $\mathrm{Im}[\zeta_e Z(\zeta_e)] \approx \sqrt{\pi}\,\zeta_e$ to leading order, which drops the exponential and gives only a linear dependence on the drift. The large-argument expansion of $Z(\zeta)$, on the other hand, retains the factor: $\mathrm{Im}[\zeta_e Z(\zeta_e)] \approx \sqrt{\pi}\,\zeta_e\, e^{-\zeta_e^2}$. But the crucial point is that *both* are approximations to the **same exact result** $\sqrt{\pi}\,\zeta_e\, e^{-\zeta_e^2}$. The only difference is that the small-argument expansion additionally takes $e^{-\zeta_e^2} \approx 1$, which is valid when $|\zeta_e| \ll 1$ but breaks down when the drift pushes ζ_e away from zero. The exponential is therefore *always* present in the exact theory; it is not an artifact of choosing one expansion over the other. When there is no drift ($v_{oe} = 0$), $\zeta_e = \omega_r/(\sqrt{2}\,k\,v_{\mathrm{th},e}) \ll 1$, and $e^{-\zeta_e^2} \approx 1$ is a perfectly good approximation. But with a drift, $(\omega_r - \mathbf{k}\cdot\mathbf{v}_{oe})$ can become comparable to $k\,v_{\mathrm{th},e}$, and the exponential matters — you cannot drop it.

We therefore keep the exact form for the electron imaginary contribution:

$$\begin{aligned}\mathrm{Im}[K_e] &= \frac{k_e^2}{k^2}\,\sqrt{\pi}\,\zeta_e\, e^{-\zeta_e^2} \\ &= \sqrt{\frac{\pi}{2}}\;\frac{k_e^2}{k^2}\;\frac{\omega_r - \mathbf{k}\cdot\mathbf{v}_{oe}}{k\,v_{\mathrm{th},e}}\;\exp\!\Big[-\frac{(\omega_r - \mathbf{k}\cdot\mathbf{v}_{oe})^2}{2k^2 v_{\mathrm{th},e}^2}\Big].\end{aligned} \tag{4.2.11}$$

The ion imaginary contribution is exponentially suppressed ($e^{-\zeta_i^2}$ with $\zeta_i \gg 1$), so $D_i \approx \mathrm{Im}[K_e]$.

Assembling the growth rate. Substituting into Equation (4.2.5):

$$\begin{aligned}\gamma &= -\sqrt{\frac{\pi}{2}}\;\frac{k_e^2}{k^2}\;\frac{\omega_r - \mathbf{k}\cdot\mathbf{v}_{oe}}{k\,v_{\mathrm{th},e}}\;e^{-\zeta_e^2}\;\Big/\;\frac{2k_e^2}{k^2\,\omega_r} \\ &= \sqrt{\frac{\pi}{8}}\;\frac{\omega_r\,(\mathbf{k}\cdot\mathbf{v}_{oe} - \omega_r)}{k\,v_{\mathrm{th},e}^2}\;\exp\!\Big[-\frac{(\omega_r - \mathbf{k}\cdot\mathbf{v}_{oe})^2}{2k^2 v_{\mathrm{th},e}^2}\Big].\end{aligned} \tag{4.2.12}$$

Using the ion-acoustic dispersion $\omega_r \approx kc_s$ and $c_s^2 = T_e/m_i$, this can be rearranged into the standard form:

$$\boxed{\gamma \approx \sqrt{\frac{\pi}{8}}\;\frac{\omega_r^3}{k^3}\,\frac{m_i}{T_e}\;\frac{\mathbf{k}\cdot\mathbf{v}_{oe} - \omega_r}{v_{\mathrm{th},e}}\;\exp\!\Big[-\frac{(\omega_r - \mathbf{k}\cdot\mathbf{v}_{oe})^2}{2k^2 v_{\mathrm{th},e}^2}\Big].} \tag{4.2.13}$$

A positive growth rate ($\gamma > 0$) requires $\mathbf{k}\cdot\mathbf{v}_{oe} > \omega_r$ — the electron drift projected along $\mathbf{k}$ must exceed the wave phase speed. Physically, this is inverse Landau damping: the drifting electron distribution has a positive slope ($\partial f_e/\partial v > 0$) at the resonant velocity, so more electrons are pulled *backward* by the wave (losing energy) than are pulled forward, and the wave *grows*.

Here we havve derived the growth rate again in really detail, good job!

A Evaluation of $K_j^{(M)}$ for a Maxwellian Plasma

Starting from Equation (3.3.17), the quantity we need to evaluate is the velocity-space integral:

$$\mathcal{I}_j(\mathbf{k},\omega) \equiv \frac{i}{n_j}\int d^3v\, f_j^{(M)}(v)\int_{-\infty}^{0} d\tau\, \exp\bigl[ib_j(\tau,\omega)\bigr]. \tag{A.0.1}$$

Recall that $b_j(\tau,\omega) = \frac{k_\perp v_\perp}{\Omega_j}\bigl[\cos(\Omega_j\tau-\phi)-\cos\phi\bigr] + (k_z v_z-\omega)\tau$ and we have set $k_\perp = k_y$ without loss of generality by rotating the coordinate system. P.S. in this little section, we evaluate the integral of the type:

$$A_j(\mathbf{k},\omega) \equiv \frac{i}{n_j}\int d^3v f_j^{(M)}(v)\int_{-\infty}^{0} d\tau \exp[ib_j(\tau,\omega)]. \tag{A.0.2}$$

We have:

$$\exp[ib_j(\tau,\omega)] = \exp\left[i\frac{k_y v_\perp}{\Omega_j}\cos(\Omega_j\tau-\phi)\right]\cdot\exp\left[-i\frac{k_y v_\perp}{\Omega_j}\cos\phi\right]\cdot\exp[i(k_z v_z-\omega)\tau]. \tag{A.0.3}$$

We also have the Jacobi-Anger expansion:

$$\exp(izcos\Phi) = \sum_{m=-\infty}^{\infty} i^m \exp(im\Phi)J_m(z). \tag{A.0.4}$$

Let expand the items: first one:

$$\exp[iz\cos(\Omega_j\tau-\phi)] = \sum_{m=-\infty}^{\infty} i^m e^{im(\Omega_j\tau-\phi)}J_m(z); \tag{A.0.5}$$

second one:

$$\exp[-iz\cos\phi] = \sum_{n=-\infty}^{\infty}(-i)^n e^{in\phi}J_n(z). \tag{A.0.6}$$

Combining these two, we have:

$$\exp[ib_j(\tau,\omega)] = \sum_{m,n=-\infty}^{\infty} i^{m-n}e^{i(n-m)\phi}J_m(z)J_n(z)e^{i(k_z v_z-\omega+m\Omega_j)\tau} \tag{A.0.7}$$

Integrate over τ from $-\infty$ to 0, we have:

$$\int_{-\infty}^{0} d\tau e^{i(k_z v_z-\omega+m\Omega_j)\tau} = \left[\frac{e^{i(k_z v_z-\omega+m\Omega_j)\tau}}{i(k_z v_z-\omega+m\Omega_j)}\right]_{-\infty}^{0} = \frac{1}{i(k_z v_z-\omega+m\Omega_j)}. \tag{A.0.8}$$

Then we have:

$$\int_{-\infty}^{0} d\tau \exp[ib_j(\tau,\omega)] = \sum_{m,n=-\infty}^{\infty}\frac{i^{m-n}e^{i(m-n)\phi}J_m(z)J_n(z)}{i(k_z v_z-\omega-m\Omega_j)}. \tag{A.0.9}$$

Since a Maxwellian $f_j^{(M)}$ is independent of ϕ, we first integrate over ϕ from 0 to 2π:

$$\int_0^{2\pi} e^{i(m-n)\phi} d\phi = 2\pi\delta_{mn}, \tag{A.0.10}$$

this actuall quite obvious that only the terms with $m = n$ survive. So we have:

$$\int_0^{2\pi} d\phi \int_{-\infty}^{0} d\tau \exp[ib_j(\tau,\omega)] = \sum_{m=-\infty}^{\infty} \frac{2\pi J_m^2(k_y v_\perp/\Omega_j)}{i(k_z v_z - \omega - m\Omega_j)}. \tag{A.0.11}$$

In Maxwellian $f_j^{(M)} = \frac{n_j}{(\pi v_{th,j}^2)^{3/2}} \exp\left(-\frac{v^2}{v_{th,j}^2}\right)$, $d^3v = v_\perp dv_\perp d\phi dv_z$. For $v_\perp$ integral part, this is what involved:

$$\int_0^{\infty} dv_\perp v_\perp \exp\left(-\frac{v_\perp^2}{v_j^2}\right) J_m^2\left(\frac{k_y v_\perp}{\Omega_j}\right), \tag{A.0.12}$$

We have a know relationship:

$$\int_0^{\infty} dxx \exp(-\rho^2 x^2) J_p(\alpha x) J_p(\beta x) = \frac{1}{2\rho^2} \exp\left(-\frac{\alpha^2+\beta^2}{4\rho^2}\right) I_p\left(\frac{\alpha\beta}{2\rho^2}\right), \tag{A.0.13}$$

which simplified the integral to:

$$\frac{v_j^2}{2} e^{-\lambda_j} I_m(\lambda_j), \qquad \lambda_j \equiv \frac{1}{2}\left(\frac{k_\perp v_{th,j}}{\Omega_j}\right)^2 = \frac{1}{2} k_\perp^2 \rho_j^2. \tag{A.0.14}$$

Now, to handle the v_z integral properly we need the Landau contour — exactly the same logic as the unmagnetized case, but with the pole shifted by the cyclotron harmonic $m\Omega_j$. Let $t = v_z/v_j$, so $dv_z = v_j\, dt$, and define the dimensionless resonant velocity:

$$\zeta_j^m \equiv \frac{\omega + m\Omega_j}{k_z v_j}. \tag{A.0.15}$$

Then:

$$\frac{1}{\sqrt{\pi}\, v_j} \int_{-\infty}^{\infty} dv_z\, \frac{e^{-v_z^2/v_j^2}}{k_z v_z - \omega - m\Omega_j} = \frac{1}{\sqrt{\pi}\, k_z v_j} \int_{-\infty}^{\infty} dt\, \frac{e^{-t^2}}{t - \zeta_j^m}. \tag{A.0.16}$$

The integral on the right-hand side is **exactly** the plasma dispersion function $Z(\zeta_j^m)$ — the same $Z(\zeta)$ we defined back in the unmagnetized section (see Equation (3.2.8) and the box on p. 6). For $\mathrm{Im}(\zeta_j^m) > 0$ it's the standard Hilbert transform; for $\mathrm{Im}(\zeta_j^m) \le 0$ we must analytically continue *under* the pole via the Landau prescription, which is what the Z-function already encodes.

There is one subtlety: the sign of k_z matters because it flips the sign of $\mathrm{Im}(\zeta_j^m) = \mathrm{Im}(\omega)/(k_z v_j)$. For $k_z > 0$, the causal $\mathrm{Im}(\omega) > 0$ puts the pole in the upper half-plane, and the integral gives $Z(\zeta_j^m)/(k_z v_j)$. For $k_z < 0$, the pole is in the lower half-plane; one can either use $Z(-\zeta) = 2i\sqrt{\pi}\, e^{-\zeta^2} - Z(\zeta)$ or, more simply, absorb the sign into the definition by using $|k_z|$:

$$\frac{1}{\sqrt{\pi}\, v_j} \int_{-\infty}^{\infty} dv_z\, \frac{e^{-v_z^2/v_j^2}}{k_z v_z - \omega - m\Omega_j} = \frac{1}{|k_z| v_j} Z(\zeta_j^m), \qquad \zeta_j^m \equiv \frac{\omega + m\Omega_j}{|k_z| v_j}. \tag{A.0.17}$$

In terms of the thermal speed $v_{th,j} = \sqrt{k_B T_j / m_j}$ (with $v_j = \sqrt{2}\, v_{th,j}$), this is:

$$\zeta_j^m = \frac{\omega + m\Omega_j}{\sqrt{2}\,|k_z|\, v_{th,j}}. \tag{A.0.18}$$

Assembling everything. We now have all three pieces of the velocity integral:

- ϕ-integral: $\int_0^{2\pi} d\phi\, e^{i(m-n)\phi} = 2\pi\, \delta_{mn} \;\rightarrow$ collapses the double Bessel sum to a single sum.
- $v_\perp$-integral: $\int_0^\infty dv_\perp v_\perp e^{-v_\perp^2/v_j^2} J_m^2 = \frac{v_j^2}{2}\, e^{-\lambda_j} I_m(\lambda_j)$.
- v_z-integral: $\frac{1}{\sqrt{\pi} v_j} \int dv_z \frac{e^{-v_z^2/v_j^2}}{k_z v_z - \omega - m\Omega_j} = \frac{1}{|k_z| v_j} Z(\zeta_j^m)$.

Multiplying these together and inserting the Maxwellian normalization $n_j/(\pi v_j^2)^{3/2}$, the full velocity-space integral inside $K_j^{(M)}$ becomes:

$$\begin{aligned}\frac{1}{n_j} \int d^3v\, f_j^{(M)}(v) \int_{-\infty}^{0} d\tau\, e^{i b_j(\tau,\omega)} &= \frac{1}{(\pi v_j^2)^{3/2}} \cdot 2\pi \cdot \frac{v_j^2}{2} \cdot \frac{1}{|k_z| v_j}\; e^{-\lambda_j} \sum_{m=-\infty}^{\infty} I_m(\lambda_j)\, Z(\zeta_j^m) \\ &= \frac{e^{-\lambda_j}}{|k_z| v_j} \sum_{m=-\infty}^{\infty} I_m(\lambda_j)\, Z(\zeta_j^m).\end{aligned} \tag{A.0.19}$$

Substituting this result back into Equation (3.3.17), i.e. $K_j^{(M)} = \frac{k_j^2}{k^2}\left[1 + \omega\, \mathcal{I}_j\right]$, yields the final susceptibility quoted in the main text.