# Basic MHD Equations

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This is the lecture notes for Magnetohydrodymacis (MHD) equations, including the physical discussions and equations derivations.

# 1 Equations and Derivations

## 1.1 Electromagnetic Equations

**Maxwell's Equations**

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{j} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \tag{1.1}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1.2}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{1.3}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}, \tag{1.4}$$

where $\boldsymbol{E}$ is the *electric field*, $\boldsymbol{B}$ is the *magnetic field*, $\boldsymbol{j}$ is the *current density*, $\rho$ is the *charge density*, $\mu$ is the *magnetic permeability*, $\epsilon_0$ is the *permittivity of free space*, and $c$ is the *speed of light* in vacuum ($\sim 2.998 \times 10^8$ m/s). The four Maxwell equations represent different physical phenomena, the first equation suggests that either currents or time-varying electric fields can generate magnetic fields, the second equation indicates that there are no magnetic monopoles and a magnetic flux tube has a constant strength along its length, and the third and fourth equations imply that either time-varying magnetic fields or electric charges can give rise to electric fields.

Here we also introduce the concept of *Debye Length*, which is a measure of a charge carrier's net electrostatic effect in a solution and how far its electrostatic effect perssists. The potential function for a single electron is

$$\Phi(r) = \frac{e}{4\pi\epsilon_0 r}.$$

Under the influence of a positive charge q, the charge distribution satisfies Boltzmann's distribution:

$$n_e = n_{e0} exp(-\frac{e\Phi(r)}{k_B T_e}) \approx n_{e0}(1 + \frac{e\Phi(r)}{k_B T_e})$$

Assume $n_e = n_{e0} + \delta n_e$, we have:

$$\delta n_e = n_{e0}\frac{e\Phi(r)}{k_B T_e}.$$

Possion's equation claims:

$$\nabla^2 \Phi(r) = -\frac{\rho}{\epsilon_0} = \frac{e^2 n_{e0}}{k_B T_e \epsilon_0}\Phi(r).$$

Combining the boundary conditions $\Phi(r \to \inf) = 0$ and $\Phi(r \to 0) = \frac{q}{4\pi\epsilon_0 r}$, we have the Debye potential function $\Phi$:

$$\Phi(r) = \frac{q}{4\pi\epsilon_0 r} exp(-\frac{r}{\lambda_D}), \tag{1.5}$$

where

$$\lambda_D = \sqrt{\frac{k_B T_e \epsilon_0}{e^2 n_{e0}}} \tag{1.6}$$

is the *Debye Length*.

### Generalized Ohm's Law

*Ohm's Law* asserts that the current density is proportional to the total electric field (in a frame of reference moving with the frame), it can be written as:

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}), \tag{1.7}$$

where $\sigma$ is the electrical conductivity.

However, a generalization of Ohm's Law may be more appropriate in some regions of the space. We start from equations of motion for electrons and ions (assumed to be protons):

$$\frac{\partial}{\partial t}(n_e m_e \boldsymbol{u_e}) + \nabla \cdot (n_e m_e \boldsymbol{u_e} \boldsymbol{u_e}) = -\nabla \cdot \boldsymbol{P_e} - e n_e(\boldsymbol{E} + \boldsymbol{u_e} \times \boldsymbol{B}) + \boldsymbol{R_e},$$

$$\frac{\partial}{\partial t}(n_i m_i \boldsymbol{u_i}) + \nabla \cdot (n_i m_i \boldsymbol{u_i} \boldsymbol{u_i}) = -\nabla \cdot \boldsymbol{P_i} + e n_i(\boldsymbol{E} + \boldsymbol{u_i} \times \boldsymbol{B}) + \boldsymbol{R_i}.$$

Multiply the ion equation by $m_e/m_i$, then substract the electron equation from the ion equation, we have:

$$m_e \frac{\partial}{\partial t}(n_i \boldsymbol{u_i} - n_e \boldsymbol{u_e}) + m_e \nabla \cdot (n_i \boldsymbol{u_i} \boldsymbol{u_i} - n_e \boldsymbol{u_e} \boldsymbol{u_e}) =$$

$$\nabla \cdot (\boldsymbol{P_e} - \frac{m_e}{m_i}\boldsymbol{P_i}) + e[(n_e + \frac{m_e}{m_i} n_i)\boldsymbol{E} + (n_e \boldsymbol{u_e} + \frac{m_e}{m_i} n_i \boldsymbol{u_i}) \times \boldsymbol{B}] - (\boldsymbol{R_e} - \frac{m_e}{m_i}\boldsymbol{R_i}),$$

where $\boldsymbol{R_i} = -\boldsymbol{R_e}$. Note that in MHD regime we have $n_e \approx n_i \approx n$, and we neglect all the $O(m_e/m_i)$ terms, we have:

$$\frac{m_e}{e}\frac{\partial \boldsymbol{j}}{\partial t} + m_e \nabla \cdot [n(\boldsymbol{u_i} \boldsymbol{u_i} - \boldsymbol{u_e} \boldsymbol{u_e})] = \nabla \cdot \boldsymbol{P_e} + ne(\boldsymbol{E} + \boldsymbol{u_e} \times \boldsymbol{B}) - \boldsymbol{R_e}.$$

Replace $u_e$ with $u_e = u_i - \frac{\boldsymbol{j}}{ne} \approx \boldsymbol{u} - \frac{\boldsymbol{j}}{ne}$, where $u$ is the plasma velocity, we have:

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \frac{m_e}{ne^2}\frac{\partial \boldsymbol{j}}{\partial t} - \frac{1}{ne}\nabla \cdot \boldsymbol{P_e} + \frac{\boldsymbol{j} \times \boldsymbol{B}}{ne} + \frac{1}{ne}\boldsymbol{R_e} + \frac{m_e}{ne^2}\nabla \cdot (\boldsymbol{j}\boldsymbol{u} + \boldsymbol{u}\boldsymbol{j} - \frac{\boldsymbol{j}\boldsymbol{j}}{ne}).$$

$R_e$ represents the rate of change of the electron momentum due to the collision with ion, which can be written as:

$$\boldsymbol{R_e} = n m_e \mu_c(\boldsymbol{u_i} - \boldsymbol{u_e}) = \frac{\mu_c m_e}{e}\boldsymbol{j},$$

where $\mu_c$ is the collision frequency. By defining the resistivity $\eta = \frac{\mu_c m_e}{ne^2}$, we can write the collision term as

$$\frac{\boldsymbol{R_e}}{ne} = \eta \boldsymbol{j}.$$

Finally, we can write the *generalized Ohm's Law* as:

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j} + \frac{1}{ne}\boldsymbol{j} \times \boldsymbol{B} - \frac{1}{ne}\nabla \cdot \boldsymbol{P_e} + \frac{m_e}{ne^2}[\frac{\partial \boldsymbol{j}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{j} + \boldsymbol{j}\boldsymbol{u} - \frac{\boldsymbol{j}\boldsymbol{j}}{ne})]. \tag{1.8}$$

### Induction Equation

Combining equations (1), (3), and (7), we can easily have:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (-\boldsymbol{v} \times \boldsymbol{B} + \boldsymbol{j}/\sigma) = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \nabla \times (\eta \nabla \times \boldsymbol{B}),$$

where $\eta = 1/(\mu\sigma)$ is the *magnetic diffusivity*. Identify that $\nabla \times (\nabla \times \boldsymbol{B}) = \nabla(\nabla \cdot \boldsymbol{B}) - \nabla^2 \boldsymbol{B}$, we find the *induction equation*:

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B}. \tag{1.9}$$

## 1.2 Plasma Equations

### Mass Continuity

The *equation of mass conservation* usually comes in two forms (equivalent to each other):

$$\frac{d\rho}{dt} + \rho\nabla\cdot\boldsymbol{v} = 0\ (a) \qquad or \qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0\ (b), \tag{1.10}$$

where $\rho$ is the *mass density* and $\boldsymbol{v}$ is the *velocity* of the fluid.

### Equation of Motion

Under the condition of electrical neutrality, the *equation of motion* can be written as:

$$\rho\frac{d\boldsymbol{v}}{dt} = -\nabla p + \boldsymbol{j}\times\boldsymbol{B} + \boldsymbol{F}, \tag{1.11}$$

where $p$ is the *pressure*, $\boldsymbol{j}\times\boldsymbol{B}$ is the *Lorentz force*, and $\boldsymbol{F}$ represents the *external force*. Usually, the external force consists of two parts: the *gravity* ($\boldsymbol{F_g}$) and *viscosity* ($\boldsymbol{F_v}$). The gravity force is:

$$F_g = -\rho g(r)\hat{r},$$

where $g(r) = M(r)G/r^2$ is the *local gravitational acceleration*. The viscous force is

$$F_v = \rho\nu[\nabla^2 + \frac{1}{3}\nabla(\nabla\cdot\boldsymbol{v})],$$

where $\nu$ is the *kinematic viscosity*. We can simplify the force to $\boldsymbol{F}_v = \rho\nu\nabla^2\boldsymbol{v}$ when the flow is incompressible.

### Perfect Gas Law

The gas pressure is determined by an equation of state, which is taken for simplicity as the *perfect gas law*:

$$p = \frac{\tilde{R}}{\tilde{\mu}}\rho T = \frac{k_B}{m}\rho T = nk_BT, \tag{1.12}$$

where $\tilde{R}$ is the *specific gas constant*, $\tilde{\mu}$ is the *mean molecular weight*, and $T$ is the *temperature*, $n$ is the *total number* of particles per volume.

## 1.3 Energy Equations

### Different Forms of the Heat Equation

The *heat equation* is the last fundamental equation, which can be written as:

$$\rho T\frac{ds}{dt} = -\mathcal{L}, \tag{1.13}$$

where $\mathcal{L}$ is the *energy loss function*, representing the net effect of all the sinks and sources of energy in the system. $s$ is the *entropy per unit mass*. The equation states that the rate of increase of heat for

a unit volume as it moves in space is due to the net effect of the energy sinks and sources. When the energy losses and gains balance, i.e. $\mathcal{L} = 0$, the entropy is conserved.

The energy equation can be defined in alternative ways. For example, in terms of the *internal energy* (e) per unit mass, it becomes:

$$\rho \frac{de}{dt} - \frac{p}{\rho}\frac{d\rho}{dt} = -\mathcal{L}.$$

For an ideal polytropic gas the internal energy is $e = c_\nu T$, where $c_\nu$ is the *specific heat at constant volume*. Furthermore, the *specific heat at constant pressure* ($c_p$) and the *ratio of specific heats* ($\gamma$) are defined by: $c_p = c_\nu + k_b/m$ and $\gamma = c_p/c_\nu$, which together claim that:

$$c_p = \frac{\gamma}{\gamma - 1}\frac{k_B}{m}, \quad , c_\nu = \frac{1}{\gamma - 1}\frac{k_B}{m}, \quad and \quad e = \frac{p}{(\gamma - 1)\rho}.$$

**Thermal Conduction**

Let us consider the terms in the energy loss function $\mathcal{L}$, which, in general, may be written as the rate of energy loss minus the rate of energy gain:

$$\mathcal{L} = \nabla \cdot \boldsymbol{q} + L_r + j^2/\sigma - F_H,$$

where $\boldsymbol{q}$ is the *heat flux* due to particle conduction, $L_r$ is the net radiation, $j^2/\sigma$ is the Ohmic dissipation, and $F_H$ represents the sum of all the other heating sources.

The heat flux vector can be written as:

$$\boldsymbol{q} = -\kappa \nabla T,$$

where $\kappa$ is the *thermal conduction tensor*. The heat flux vector can be split into two parts:

$$\nabla_\parallel \cdot (\kappa_\parallel \nabla_\parallel T) + \nabla_\perp \cdot (\kappa_\perp \nabla_\perp T),$$

where $\parallel$ and $\perp$ denote directions parallel and perpendicular to the magnetic field. For a fully ionized plasma, we have:

$$\kappa_\parallel = \kappa_0 T^{5/2} = 1.8 \times 10^{-10}\frac{T^{5/2}}{In\Lambda} \quad Wm^{-1}K^{-1}.$$

Typical values are: $4 \times 10^{-11}T^{5/2}$ for photosphere, $10^{-11}T^{5/2}$ for chromosphere, and $9 \times 10^{-12}T^{5/2}$ for corona. Spitzer gives that $\kappa_\perp/\kappa_\parallel = 2 \times 10^{-31}n^2/(T^3B^2)$ with B in T. In a magnetic that is strong enough to make $\kappa_\perp \ll \kappa_\parallel$, we can ignore the parallel item and the heat conduction term approximates $\nabla_\parallel \cdot (\kappa_\parallel \nabla_\parallel T)$.

**Radiation**

In the solar interior, energy transport by radiation dominates particle conduction. The net radiation may be written as the divergence of a radiative flux ($\boldsymbol{q}_r$), namely, $L_r = \nabla \cdot \boldsymbol{q}_r$, where $\boldsymbol{q}_r = -\kappa_r \nabla T$ and $\kappa_r = 16\sigma_s T^3/(3\widetilde{\kappa}\rho)$ is the coefficient of radiative conductivity. Here, $\sigma_s$ is the *Stefan-Boltzmann constant*, $\widetilde{\kappa}$ is the *opacity* or *mass absorption coefficient* and $\widetilde{\kappa}$ is the *absorption coefficient*. If we consider $\kappa_r$ to be locally uniform, then the radiative loss reduces to the simple form $L_r = -\kappa_r \nabla^2 T$.

## 1.4 Vlasov's Equation

Let us first consider the physical interpretation of the *Vlasov equation*:

$$\frac{\partial}{\partial t}f + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f + \frac{q}{m}(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}}f = 0. \tag{1.14}$$

We can write it in a total time derivative form:

$$\frac{d}{dt}f(\boldsymbol{x},\boldsymbol{v},t) = \frac{\partial}{\partial t}f + \underbrace{\frac{d\boldsymbol{x}}{dt}}_{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{x}}f + \underbrace{\frac{d\boldsymbol{v}}{dt}}_{\frac{q}{m}(\boldsymbol{E}+\frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})}\cdot\nabla_{\boldsymbol{v}}f = 0, \tag{1.15}$$

where $\frac{df}{dt}$ is the total time derivative of $f$. The total time derivative follows the orbit of a particle in the $\boldsymbol{x}-\boldsymbol{v}$ plane. It evaluates the variation of $f$ due to the change of the particle position in the phase space as well as the time change. Usually, the physical conditions in the solar wind suggest that we can neglect the effect of collisions. However, if we want to consider the effect, we can add a symbol:

$$\frac{\partial}{\partial t}f + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f + \frac{q}{m}(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}}f = (\frac{\partial}{\partial t}f)_{coll}. \tag{1.16}$$

To sum up, we write the complete set of equations, which are self-consistent and coupled particle and fields:

$$\frac{\partial}{\partial t}f + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}f + \frac{q}{m}(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B})\cdot\nabla_{\boldsymbol{v}}f = 0 \tag{1.17}$$

$$\nabla\cdot\boldsymbol{B} = 0 \tag{1.18}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\rho \tag{1.19}$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \tag{1.20}$$

$$\nabla\times\boldsymbol{B} = \mu\boldsymbol{j} + \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}, \tag{1.21}$$

where:

$$\rho(\boldsymbol{x},t) = \sum_\alpha q_\alpha \int f_\alpha d^3v \tag{1.22}$$

$$\boldsymbol{J}(\boldsymbol{x},t) = \sum_\alpha q_\alpha \int \boldsymbol{v} f_\alpha d^3v. \tag{1.23}$$

0th order solution suggests that all items of LHS (eq. 1.14) are 0. Let us begin with the 1st order solution.

### 1.4.1 1st-order (Linear Theory) of Electrostatic Waves by the Normal-Mode Method:

Consider small derivations from the equilibrium of electrostatic waves ($f = f_0(v)$, $\boldsymbol{E}_0 = \boldsymbol{B} = 0$, and no external magnetic field), we may write:

$$\begin{aligned} f(\boldsymbol{x},\boldsymbol{v},t) &= f_0(\boldsymbol{v}) + f_1(\boldsymbol{x},\boldsymbol{v},t) \\ \phi(\boldsymbol{x},t) &= \phi_1(\boldsymbol{x},t) \\ \boldsymbol{E} &= \boldsymbol{E}_1(\boldsymbol{x},t) \end{aligned} \tag{1.24}$$

Sub eq. 1.24 into eq. 1.14, we have:

$$\frac{\partial}{\partial t} f_1 + \boldsymbol{v} \cdot \nabla f_1 = -\frac{e}{m} \nabla \phi_1 \cdot \nabla_{\boldsymbol{v}} f. \tag{1.25}$$

Note that $\boldsymbol{E}_1 = -\nabla \phi_1$, $\rho_1 = -e \int f_1 d^3 v$, and $\nabla \cdot \boldsymbol{E_1} = 4\pi \rho_1$, we have

$$\nabla^2 \phi_1 = 4\pi e \int f_1 d^3 v. \tag{1.26}$$

Consider a normal mode in the complex space:

$$\begin{aligned} f_1(\boldsymbol{x}, \boldsymbol{v}, t) &= f_{1k}(\boldsymbol{v}) e^{ik_z z - i\omega t} \\ \phi_1(\boldsymbol{x}, t) &= \phi_{1k} e^{ik_z z - i\omega t}. \end{aligned} \tag{1.27}$$

Eq. 1.25, 1.26 and 1.27 suggest that:

$$-k_z^2 \phi_{1k} = \underbrace{\frac{4\pi n_0 e^2}{m_e}}_{\omega_{pe}^2} \frac{1}{n_0} k_z \phi_{1k} \int \frac{\partial f_0(\boldsymbol{v})/\partial v_z}{\omega - k_z v_z} d^3 v. \tag{1.28}$$

For $\phi_{1k} \neq 0$, we must have:

$$1 - \frac{\omega_{pe}^2}{k_z^2} \frac{1}{n_0} \int_{-\infty}^{\infty} \frac{\partial f_0(\boldsymbol{v})/\partial v_z}{v_z - \omega/k_z} d^3 v = 0. \tag{1.29}$$

We use a one-dimensional distribution function $g_0(v_z) = \frac{1}{n_0} \int f_0(\boldsymbol{v}) dv_x dv_y$, which makes the equation become:

$$1 - \frac{\omega_{pe}^2}{k_z^2} \int_{-\infty}^{\infty} \frac{dg_0(v_z)/dv_z}{v_z - \omega/k_z} dv_z = 0. \tag{1.30}$$

Given that $g_0(v_z \to \pm\infty) = 0$, integrate the above equation shows:

$$1 - \frac{\omega_{pe}^2}{k - z^2} \int_{-\infty}^{+\infty} \frac{g_0(\boldsymbol{v})}{(v_z \omega/k_z)^2} dv_z = 1 - \frac{\omega_{pe}^2}{\omega^2} \int_{-\infty}^{+\infty} \frac{g_0(v_z)}{(1 - k_z v_z/\omega_z)^2} dv_z = 0. \tag{1.31}$$

Expand $\frac{1}{(1-k_z v_z/\omega_z)^2}$, we have:

$$1 - \frac{\omega_{pe}^2}{\omega^2} \int_{-\infty}^{+\infty} g_0(v_z)[1 + 2\frac{k_z v_z}{\omega} + 3(\frac{k_z v_z}{\omega})^2] dv_z = 0. \tag{1.32}$$

The Maxwellian distribution is a natural solution:

$$f_0(\boldsymbol{v}) = \frac{n_0}{(2\pi)^{3/2} v_{Te}^3} exp(-\frac{v_x^2 + v_y^2 + v_z^2}{2v_{Te}^2}). \tag{1.33}$$

Note a useful integration $\int_0^{+\infty} e^{-ax^2} dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}$ and $g_0(v_z) = \frac{1}{n_0} \int f_0(\boldsymbol{v}) dv_x dv_y$, we have;

$$g_0(v_z) = \frac{1}{\sqrt{2\pi} v_{Te}} exp(-\frac{v_z^2}{2v_{Te}^2}). \tag{1.34}$$

Note another useful integration $\int_0^{+\infty} x^2 e^{-ax^2} dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$, we have:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} g_0(v_z) dv_z = 1 \\
&\int_{-\infty}^{+\infty} g_0(v_z) v_z dv_z = 0 \\
&\int_{-\infty}^{+\infty} g_0(v_z) v_z^2 dv_z = v_{Te}^2.
\end{aligned} \tag{1.35}$$

Sub the above equations into eq. 1.32, we have:

$$1 - \frac{\omega_{pe}^2}{\omega^2}[1 + 2\frac{k_z v_{Te}}{\omega} + 3(\frac{k_z v_{Te}}{\omega})^2] = 0, \tag{1.36}$$

i.e.,

$$\omega^2 = \omega_{pe}^2 + 3k_z^2 v_{Te}^2. \tag{1.37}$$

The above equation states the dispersion relation for a Maxwellian distribution in such physical conditions.

### 1.4.2 Landau Contour

Before we dive into the Landau contour, let us first review some relevant definitions and theorems involving complex variables.

#### Laplace Transform

$$\begin{aligned}
&\mathcal{L}[\phi(t)] \equiv \int_0^{\infty} \phi(t) e^{-pt} dt = \tilde{\phi}(p) \\
&\mathcal{L}[\phi'(t)] = p\tilde{\phi}(p) - \phi(0) \\
&\mathcal{L}[\phi''(t)] = p^2\tilde{\phi}(p) - p\phi(0) - \phi'(0).
\end{aligned} \tag{1.38}$$

Inverse Laplace transform:

$$\phi(t) = \frac{1}{2\pi i}\int_{p_0 - i\infty}^{p_0 + i\infty} \tilde{\phi}(p) e^{pt} dp. \tag{1.39}$$

**Analytic Function** A function $f(z)$ in the complex z-plane ($z = x + iy = re^{i\theta}$) is said to be analytic at point z if it has a derivative there and the derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} (h : complex)$$

is independent of the path by which $h$ approaches 0. The necessary and sufficient condition for a function

$$W(z) = U(x, y) + iV(x, y)$$

to be analytic are:

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \qquad \& \qquad \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}.$$

For example, the function $f(z) = z^2$ is analytic everywhere except at $z = 0$.

**Single-Valued Function** A function is said to be single-valued if:

$$W(z = re^{i\theta}) = W(z = re^{i(\theta+2\pi n)}),$$

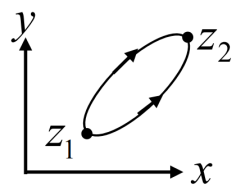
where $n$ is an integer. For example, the function $f(z) = z^2$ is single-valued, while $f(z) = \sqrt{z}$ is not single-valued.

**Regular Function** A function is said to be regular in a region R if it is both analytic and single-valued.

**Cauchy's Theorem** If a function $f(z)$ is regular in a region R, then:

$$\oint_c f(z)dz = 0,$$

where C is any closed path lying within R. Hence, the line integral $\int_{z_1}^{z_2} f(z)dz$ is independent of the path of integration from $z_1$ to $z_2$ if the path lies within region R.

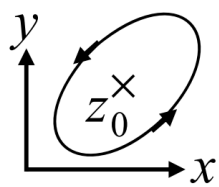



**Theorem of Residues** If $f(z)$ is regular in a region R, except for a finite number of poles $z_1, z_2, ..., z_n$, then:

$$\oint_C f(z)dz = 2\pi i \sum_{j=1}^{n} \quad residues \quad insideC,$$

where C is any closed path lying within R and enclosing the poles $z_1, z_2, ..., z_n$. The residue of a pole of order n at $z = z_0$ can be calculated as:

$$\frac{1}{(n-1)!}\{(\frac{d}{dz})^{n-1}[(z-z_0)^n f(z)]\}_{z=z_0}.$$




Now, let us calculate the dispersion relation of electrostatic waves by the method of Laplace Transform. Start from eqs. 1.25 and 1.26 (for convenience, rewrite them here):

$$\frac{\partial}{\partial t}f_1 + \boldsymbol{v}\cdot\nabla f_1 = -\frac{e}{m}\nabla\phi_1\cdot\nabla_{\boldsymbol{v}} f_1$$
$$\nabla^2\phi_1 = 4\pi e\int f_1 d^3v.$$

Landau resolved this problem by treating the above two equations as an initial value problem in t, while analyzing a spatial Fourier component in z (denoted by subscript "k"). Let

$$\begin{aligned} f_1(\boldsymbol{x},\boldsymbol{v},t) &= f_{1k}(\boldsymbol{v},t)e^{ikz} \\ \phi_1(\boldsymbol{x},t) &= \phi_{1k}(t)e^{ikz}. \end{aligned} \tag{1.40}$$

Substitute the above equations into eqs. 1.25 and 1.26, we have:

$$\begin{aligned} \frac{\partial}{\partial t}f + ik_z v_z f_{1k}(\boldsymbol{v},t) &= -\frac{e}{m_e}ik_z\phi_{1k}(t)\frac{\partial f_0(\boldsymbol{v})}{\partial v_z} \\ -k_z^2\phi_{1k}(t) &= 4\pi e\int f_{1k}(\boldsymbol{v},t)d^3v. \end{aligned} \tag{1.41}$$

Perform Laplace transform on the above two equations, we have:

$$
\begin{aligned}
&p\tilde{f}_{1k}(\boldsymbol{v},p) - f_{1k}(\boldsymbol{v},0) + ik_z v_z \tilde{f}_{1k}(\boldsymbol{v},p) = -\frac{e}{m_e} ik_z \tilde{\phi}_{1k}(p) \frac{\partial f_0(\boldsymbol{v})}{\partial v_z} \\
&- k_z^2 \tilde{\phi}_{1k}(p) = 4\pi e \int \tilde{f}_{1k}(\boldsymbol{v},p) d^3v,
\end{aligned}
\tag{1.42}
$$

where

$$
\begin{aligned}
\tilde{f}_{1k}(\boldsymbol{v},p) &= \int_0^\infty f_{1k}(\boldsymbol{v},t) e^{-pt} dt \\
\tilde{\phi}_{1k}(p) &= \int_0^\infty \phi_{1k}(t) e^{-pt} dt.
\end{aligned}
\tag{1.43}
$$

From the above equations, we can solve $\tilde{f}_{1k}(\boldsymbol{v},p)$ by:

$$
\tilde{f}_{1k}(\boldsymbol{v},p) = \frac{f_{1k}(\boldsymbol{v},0) - \frac{e}{m} ik_z \tilde{\phi}_{1k}(p) \frac{f_0(\boldsymbol{v})}{\partial v_z}}{p + ik_z v_z}.
\tag{1.44}
$$

Then, we can substitute the above equation into the second equation to solve $\tilde{\phi}_{1k}(p)$:

$$
\begin{aligned}
&\tilde{\phi}_{1k}(t) = -\frac{1}{k_z^2} 4\pi e [\int \frac{f_{1k}(\boldsymbol{v},0)}{p + ik_z v_z} d^3v - \frac{e}{m} ik_z \tilde{\phi}_{1k}(p) \int \frac{\partial f_0(\boldsymbol{v})}{\partial v_z} d^3v] \\
\Rightarrow \quad &\tilde{\phi}_{1k}(t)[1 - \frac{4\pi e^2}{k_z^2 m} ik_z \int \frac{\partial f_0(\boldsymbol{v})}{\partial v_z} d^3v] = -\frac{1}{k_z^2} 4\pi e \int \frac{f_{1k}(\boldsymbol{v},0)}{p + ik_z v_z} d^3v \\
\Rightarrow \quad &\tilde{\phi}_{1k}(t) = -\frac{-4\pi e \int_{-\infty}^{+\infty} \frac{f_{1k}(\boldsymbol{v},0)}{p+ik_z v_z} d^3v}{k_z^2 [1 - \frac{\omega_{pe}^2}{k_z^2} \int_{-\infty}^{+\infty} \frac{\frac{ik_z}{n_0} \frac{\partial f_0(\boldsymbol{v})}{\partial v_z}}{p+ik_z v_z} d^3v]} = \frac{-i \frac{4\pi n_0 e}{k_z^3} \int_{-\infty}^{+\infty} \frac{g_{1k}(v_z,0)}{v_z - \frac{ip}{k_z}} dv_z}{1 - \frac{\omega_{pe}^2}{k_z^2} \int_{-\infty}^{+\infty} \frac{\frac{dg_0(v_z)}{dv_z}}{v_z - \frac{ip}{k_z}} dv_z}
\end{aligned}
\tag{1.45}
$$

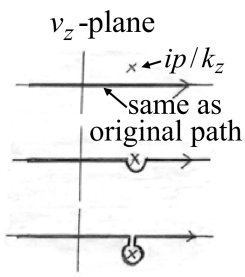
where

$$
\begin{aligned}
g_{1k}(v_z,0) &= \frac{1}{n_0} \int f_{1k}(\boldsymbol{v},0) dv_x v_y \\
g_0(v_z) &= \frac{1}{n_0} \int f_0(\boldsymbol{v}) dv_x dv_y.
\end{aligned}
\tag{1.46}
$$

With inverse Laplace transform, we can obtain the solution of $\phi_{1k}(t)$:

$$
\phi_{1k}(t) = \frac{1}{2\pi i} \int_{p_0 - i\infty}^{p_0 + i\infty} \tilde{\phi}_{1k}(p) e^{pt} dp.
\tag{1.47}
$$

Note the expression of $\tilde{\phi}_{1k}(t)$, we can see that there is a pole existing at $p = -ik_z v_z$. This means we might need to change the path of the original straight line $v_z$ integration to the Landau contour: $\int_L dv_z$.



Landau contour

### 1.4.3 General Theory of Linear Vlasov Waves

Now, let us take a further step. Instead of a zero-external-field assumption, we consider an external magnetic field $\boldsymbol{B} = B_0\boldsymbol{e}_z$. For 0 th order solution, the Vlasov equation

$$\underbrace{\frac{\partial}{\partial t}f_{\alpha 0}(\boldsymbol{v})}_{0} + \boldsymbol{v}\cdot\underbrace{\nabla f_{\alpha 0}(\boldsymbol{v})}_{0} - \frac{q}{m_\alpha}(\underbrace{\boldsymbol{E}_0}_{0} + \frac{1}{c}\boldsymbol{v}\times B_0\boldsymbol{e}_z)\cdot\nabla_v f_{\alpha 0}(\boldsymbol{v}) = 0$$

becomes:

$$(\boldsymbol{v}\times B_0\boldsymbol{e}_z)\cdot\nabla_v f_{\alpha 0}(\boldsymbol{v}) = 0. \tag{1.48}$$

Examples of equilibrium solutions are:

$$\begin{aligned} f_0 &= \frac{n_0}{(2\pi)^{3/2}v_T^3}exp(-\frac{v_x^2+v_y^2+v_z^2}{2v_T^2}) \qquad [Maxwellian] \\ f_0 &= \frac{n_0}{(2\pi)^{3/2}v_{T\perp}^2 v_{Tz}}exp(-\frac{v_\perp^2}{2v_{T\perp}^2} - \frac{v_z^2}{2v_{Tz}^2}) \qquad [Bi-Maxwellian]. \end{aligned} \tag{1.49}$$

Then we move to 1 st order solution. To obtain a dispersion relation, we first linearize the set of plasma equations.

$$\begin{aligned} &f_\alpha(\boldsymbol{x},\boldsymbol{v},t) = f_{\alpha 0}(\boldsymbol{v}) + f_{\alpha 1}(\boldsymbol{x},\boldsymbol{v},t), \\ &\boldsymbol{E} = \boldsymbol{E}_1(\boldsymbol{x},t), \\ &\boldsymbol{B} = B_0\boldsymbol{e}_z + \boldsymbol{B}_1(\boldsymbol{x},t), \\ &\rho(\boldsymbol{x},t) = \rho_1(\boldsymbol{x},t), \\ &\boldsymbol{J} = \boldsymbol{J}_1(\boldsymbol{x},t). \end{aligned} \tag{1.50}$$

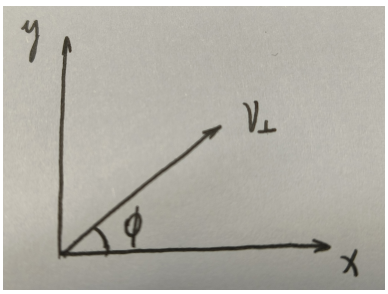
We substitute the above equation into the Vlasov'set of equations, we have:

$$\begin{aligned} &\frac{\partial}{\partial t}f_{\alpha 1} + \boldsymbol{v}\cdot\nabla f_{\alpha 1} + \frac{q_\alpha}{m_\alpha c}(\boldsymbol{v}\times B_0\boldsymbol{e}_z)\cdot\nabla_{\boldsymbol{v}} f_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}(\boldsymbol{E_1} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B_1})\cdot_{\boldsymbol{v}} f_{\alpha 0}, \\ &\nabla\cdot\boldsymbol{B}_1 = 0, \quad \nabla\cdot\boldsymbol{E}_1 = 4\pi\rho_1, \\ &\nabla\times\boldsymbol{E}_1 = -\frac{1}{c}\frac{\partial}{\partial t}B_1, \quad \nabla\times\boldsymbol{B}_1 = \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{E}_1 + \frac{4\pi}{c}\boldsymbol{J}_1, \end{aligned} \tag{1.51}$$

in which $\rho_1(\boldsymbol{x},t) = \sum_\alpha \int f_{\alpha 1}(\boldsymbol{x},\boldsymbol{v},t)d^3v$ and $\boldsymbol{J}_1(\boldsymbol{x},t) = \sum_\alpha q_\alpha \int f_{\alpha 1}(\boldsymbol{x},\boldsymbol{v},t)\boldsymbol{v}d^3v$. We first write the first equation of eqs. 1.51 in a total time derivative form:

$$\frac{d}{dt}f_{\alpha 1} = -\frac{q_\alpha}{m_\alpha}, \tag{1.52}$$

where $\frac{d}{dt}$ follows the 0-order orbit of a particle, which we denote by $\boldsymbol{x}'(t')$ and $\boldsymbol{y}'(t')$. Under the conditions: $\boldsymbol{x}'(t'=t) = \boldsymbol{x}$ and $\boldsymbol{v}'(t'=t) = \boldsymbol{v}$, we have:



$$
\begin{aligned}
v'_x(t') &= v_\perp cos[\phi - \Omega_\alpha(t'-t)] \\
v'_y(t') &= v_\perp sin[\phi - \Omega_\alpha(t'-t)] \\
v'_z(t') &= v_z \\
x'(t') &= x - \frac{v_\perp}{\Omega_\alpha} sin[\phi - \Omega_\alpha(t'-t)] + \frac{v_\perp}{\Omega_\alpha} sin\phi \\
y'(t') &= y + \frac{v_\perp}{\Omega_\alpha} cos[\phi - \Omega_\alpha(t'-t)] - \frac{v_\perp}{\Omega_\alpha} cos\phi, \\
z'(t') &= z + v_z(t'-t),
\end{aligned}
\tag{1.53}
$$

where $\Omega_\alpha = \frac{q_\alpha B_0}{m_{\alpha c}}$. Now we insert a normal mode:

$$
\begin{aligned}
f_{\alpha 1}(\boldsymbol{x}, \boldsymbol{v}, t) &= f_{\alpha 1k}(\boldsymbol{v}) e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \\
\boldsymbol{E}_1(\boldsymbol{x}, t) &= \boldsymbol{E}_{1k} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \\
\boldsymbol{B}_1(\boldsymbol{x}, t) &= \boldsymbol{B}_{1k} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \\
\boldsymbol{J}_1(\boldsymbol{x}, t) &= \boldsymbol{J}_{1k} e^{i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} = \sum_\alpha q_\alpha \int f_{\alpha 1k}(\boldsymbol{v}) \boldsymbol{v} d^3 v.
\end{aligned}
\tag{1.54}
$$

Note we did not include density $\rho$ in this set of equations. That is because we could replace $\rho$ by $\boldsymbol{J}$ in the linearized equations $\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0 \Rightarrow i\boldsymbol{k} \cdot \boldsymbol{J}_{1k} - i\omega\rho_{1k} = 0 \Rightarrow \rho_{1k} = \frac{1}{\omega}\boldsymbol{k} \cdot \boldsymbol{J}_{1k}$. Insert the normal mode into eq. 1.52, we have:

$$
\begin{aligned}
& f_{\alpha 1}(\boldsymbol{x}, \boldsymbol{v}, t) - f_{\alpha 1}[\boldsymbol{x}'(t'), \boldsymbol{v}'(t'), t']_{t'=\infty} \\
&= -\frac{q_\alpha}{m_\alpha} \int_{-\infty}^{t} dt' \{ bmE_1[\boldsymbol{x}'(t'), t'] + \frac{1}{c}\boldsymbol{v}'(t') \times \boldsymbol{B}_1[\boldsymbol{x}'(t'), t'] \} \cdot \nabla_{\boldsymbol{v}'} f_{\alpha 0}(\boldsymbol{v}'),
\end{aligned}
\tag{1.55}
$$

i.e.,

$$
\begin{aligned}
& f_{\alpha 1k}(\boldsymbol{v}) e^{i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} - f_{\alpha 1k}[\boldsymbol{v}'](t') e^{i\omega t' + i\boldsymbol{k}\cdot\boldsymbol{x}}|_{t'=-\infty} \\
&= -\frac{q_\alpha}{m_\alpha} \int_{-\infty}^{t} dt' [\boldsymbol{E}_{1k} + \frac{1}{c}\boldsymbol{v}'(t') \times \boldsymbol{B}_{1k}] e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \cdot \nabla_{\boldsymbol{v}'} f_{\alpha 0}(\boldsymbol{v}') \\
&= -\frac{q_\alpha}{m_\alpha} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \int_{-\infty}^{t} dt' [\boldsymbol{E}_{1k} + \frac{1}{c}\boldsymbol{v}'(t') \times \boldsymbol{B}_{1k}] e^{-i\omega t' - t + i\boldsymbol{k}\cdot(\boldsymbol{x}' - \boldsymbol{x})} \cdot \nabla_{\boldsymbol{v}'} f_{\alpha 0}(\boldsymbol{v}').
\end{aligned}
\tag{1.56}
$$

Assume $\omega_i (= Im\omega) > 0$, then $f_{\alpha 1k}[\boldsymbol{v}'t'] e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}|_{t'=-\infty} = 0$, the above equation becomes:

$$
f_{\alpha 1k}(\boldsymbol{v}) = -\frac{q_\alpha}{m_\alpha} e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}} \int_{-\infty}^{t} dt' [\boldsymbol{E}_{1k} + \frac{1}{c}\boldsymbol{v}'(t') \times \boldsymbol{B}_{1k}] \cdot \nabla_{\boldsymbol{v}} \cdot f_{\alpha 0}(\boldsymbol{v}') e^{-i\omega(t'-t) + i\boldsymbol{k}\cdot(\boldsymbol{x}' - \boldsymbol{x})}. \tag{1.57}
$$

Factoring out the $e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}$ dependence, we obtain:

$$
f_{\alpha 1k}(\boldsymbol{v}) = -\frac{q_\alpha}{m_\alpha} \int_{-\infty}^{t} dt' [\boldsymbol{E}_{1k} + \frac{1}{c}\boldsymbol{v}'(t') \times \boldsymbol{B}_{1k}] \cdot \nabla_{\boldsymbol{v}} \cdot f_{\alpha 0}(\boldsymbol{v}') e^{-i\omega(t'-t) + i\boldsymbol{k}\cdot(\boldsymbol{x}' - \boldsymbol{x})}. \tag{1.58}
$$

The method we use to obtain eq. 1.58 is called the *method of characteristics*.

Now we add the field equations into the linearized equations. From

$$\begin{aligned}\nabla \times \boldsymbol{E}_1 &= -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}_1,\\ \nabla \times \boldsymbol{B}_1 &= \frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{E}_1 + \frac{4\pi}{c}\boldsymbol{J}_1,\end{aligned} \tag{1.59}$$

we have:

$$\nabla \times (\nabla \times \boldsymbol{E}_1) = -\frac{1}{c}\frac{\partial}{\partial t}\nabla \times \boldsymbol{B}_1 = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\boldsymbol{E} - \frac{4\pi}{c^2}\frac{\partial}{\partial t}\boldsymbol{J}_1. \tag{1.60}$$

Assume the absence of an external magnetic field ($\Omega_\alpha = 0$), then we have $\boldsymbol{v}' = \boldsymbol{v} = const$ and $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{v}(t' - t)$, and eq. 1.58 can be written:

$$f_{\alpha 1k}(\boldsymbol{v}) = -\frac{1_\alpha}{m_\alpha}\frac{(\boldsymbol{E}_{1k} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B}_{1k}) \cdot \nabla_{\boldsymbol{v}} f_{\alpha 0}(\boldsymbol{v})}{i(\omega - \boldsymbol{k} \cdot \boldsymbol{v})}. \tag{1.61}$$

For an isotropic distribution, we have $f_{\alpha 0}(\boldsymbol{v}) = f_{\alpha 0}(v)$. Then, $(\boldsymbol{v} \times \boldsymbol{B}_{1k}) \cdot \nabla_{\boldsymbol{v}} f_{\alpha 0}(v) = 0$. Given the absence of external magnetic fields and an isotropic distribution assumption, we have:

$$f_{\alpha 1k} = \frac{q_\alpha}{m_\alpha}\frac{\boldsymbol{E}_{1k} \cdot \nabla_{\boldsymbol{v}} f_{\alpha 0}(v)}{i(\omega - k_z v_z)}. \tag{1.62}$$

Substitute the above equation into eq. 1.54, we have:

$$\boldsymbol{J}_{1k} = \sum_\alpha q_\alpha \int f_{\alpha 1k}(\boldsymbol{v})\boldsymbol{v} d^3 v = \sum_\alpha \int \frac{q_\alpha^2}{m_\alpha}\frac{\boldsymbol{E}_{1k} \cdot \nabla_{\boldsymbol{v}} f_{\alpha 0}(v)}{i(\omega - k_z v_z)}\boldsymbol{v} d^3 v. \tag{1.63}$$

The field equations

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_{1k}) + \frac{\omega^2}{c^2}\boldsymbol{E}_{1k} = -\frac{4\pi i\omega}{c^2}\boldsymbol{J}_{1k}, \tag{1.64}$$

states:

$$-k_z^2 E_{1kz}\boldsymbol{e}_z + (k_z^2 - \frac{\omega^2}{c^2})\boldsymbol{E}_{1k} - \frac{4\pi i\omega}{c^2}\boldsymbol{J}_{1k} = 0. \tag{1.65}$$

For electrostatic waves, we can set $\boldsymbol{E}_{1k} = E_{1kz}\boldsymbol{e}_z$. Then eq. 1.63 becomes:

$$\begin{aligned}\boldsymbol{J}_{1k} &= -\sum_\alpha i\frac{q_\alpha^2}{m_\alpha^2}\int \frac{(v_x\boldsymbol{e}_x + v_y\boldsymbol{e}_y + v_z\boldsymbol{e}_z)\frac{\partial}{\partial v_z}f_{\alpha 0}(v)}{i(\omega - k_z v_z)}d^3 v\\ &= -\sum_\alpha i\frac{q_\alpha^2}{m_\alpha^2}E_{1kz}\boldsymbol{e}_z \int \frac{v_z\frac{\partial f_{\alpha 0}}{\partial v_z}}{i(\omega - k_z v_z)}d^3 v.\end{aligned} \tag{1.66}$$

Define

$$g_{\alpha 0} = \frac{1}{n_{\alpha 0}}\int_{\alpha 0}(v)dv_x dv_y, \tag{1.67}$$

then we have:

$$\frac{dg_{\alpha 1}(v_z)}{d^3 v} = \frac{1}{n_{\alpha 0}}\frac{\partial f_{\alpha 1}(v)}{\partial v_z}. \tag{1.68}$$

The current $\boldsymbol{J}_{1k}$ becomes:

$$\boldsymbol{J}_{1k} = -\sum_\alpha i\frac{q_\alpha^2}{m_\alpha^2}E_{1kz}\boldsymbol{e}_z \int_{-\infty}^{+\infty}\frac{v_z\frac{g_{\alpha 0}(v_z)}{dv_z}}{\omega - k_z v_z}dv_z. \tag{1.69}$$

We can write $\frac{v_z}{\omega - k_z v_z} = \frac{1}{k_z}(-1 + \frac{\omega}{\omega - k_z v_z})$, we have:

$$\boldsymbol{J}_{1k} = -\sum_\alpha i \frac{q_\alpha^2}{m_\alpha^2} E_{1kz} \boldsymbol{e}_z \int_{-\infty}^{+\infty} \frac{1}{k_z}(-1 + \frac{\omega}{\omega - k_z v_z}) \frac{dg_{\alpha 0}(v_z)}{dv_z} dv_z. \tag{1.70}$$

Given that $g_{\alpha 1}(v_z) = 0$ at $v_z = \pm\infty$, we have:

$$\begin{aligned} \boldsymbol{J}_{1k} &= -\sum_\alpha i \frac{q_\alpha^2 n_{\alpha 0}}{m_\alpha^2} E_{1kz} \boldsymbol{e}_z \int_{-\infty}^{+\infty} \frac{1}{k_z} \frac{\omega}{\omega - k_z v_z} \frac{dg_{\alpha 0}(v_z)}{dv_z} dv_z \\ &= \sum_\alpha i \frac{q_\alpha^2 n_{\alpha 0}}{m_\alpha^2} \frac{\omega}{k_z^2} E_{1kz} \boldsymbol{e}_z \int_{\infty}^{+\infty} \frac{\frac{dg_{\alpha 0(v_z)}}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z. \end{aligned} \tag{1.71}$$

Replace $\boldsymbol{E}_{1k}$ with $E_{1k}\boldsymbol{e}_z$, and substitute the above equation into the field equation eq. 1.65, we have:

$$1 - \sum_\alpha \frac{\omega_{p\alpha}^2}{k_z^2} \int_L \frac{\frac{dg_{\alpha 0}(v_z)}{dv_z}}{v_z - \frac{\omega}{k_z}} dv_z = 0. \tag{1.72}$$

The above equation is the dispersion relation for electrostatic waves.

Now we consider a case of electromagnetic waves, in which $\boldsymbol{E}_{1k} \perp \boldsymbol{k}(= k_z \boldsymbol{e}_z)$. So without loss of generality, we can set $\boldsymbol{E}_{1k} = E_{1ky}\boldsymbol{e}_y$. Then, eqs. 1.69 and 1.65 state:

$$\begin{aligned} &\boldsymbol{J}_{1k} = \sum_\alpha -i \frac{q_\alpha^2}{m_\alpha} E_{1ky} \int (v_x \boldsymbol{e}_x + v_y \boldsymbol{e}_y + v_z \boldsymbol{e}_z) \frac{\frac{\partial f_{\alpha 0}(v)}{\partial v_y}}{\omega - k_z v_z} d^3 v, \\ &(k_z^2 - \frac{\omega^2}{c^2}) E_{1ky} \boldsymbol{e}_y - \frac{4\pi i \omega}{c^2} \boldsymbol{J}_{1k} = 0. \end{aligned} \tag{1.73}$$

Then we have:

$$\begin{aligned} \boldsymbol{J}_{1k} &= \sum_\alpha -i \frac{q_\alpha^2}{m_\alpha} E_{1ky} \boldsymbol{e}_y \int \frac{v_y \frac{\partial f_{\alpha 0}(v)}{\partial v_y}}{\omega - k_z v_z} d^3 v \\ &= \sum_\alpha -i \frac{q_\alpha^2}{m_\alpha} E_{1ky} \boldsymbol{e}_y \int \frac{f_{\alpha 0}(v)}{\omega - k_z v_z} d^3 v \\ &= \sum_\alpha i \frac{n_{\alpha 0} q_\alpha^2}{m_\alpha} E_{1ky} \boldsymbol{e}_y \int \frac{g_{\alpha 0}(v_z)}{\omega - k_z v_z} dv_z. \end{aligned} \tag{1.74}$$

Similarly, we replace $\boldsymbol{E}_{1k}$ with $E_{1ky}\boldsymbol{e}_y$, and substitute the above equation into the field equation eq. 1.65, we have:

$$k_z^2 - \frac{\omega^2}{c^2} + \omega \sum_\alpha \omega_{p\alpha}^2 \int \frac{g_{\alpha 0}(v_z)}{\omega - k_z v_z} dv_z = 0,$$

while $\int \frac{g_{\alpha 0}(v_z)}{\omega - k_z v_z} dv_z \approx \frac{1}{\omega} \int g_{\alpha 0}(v_z) dv_z = \frac{1}{\omega}$, we obtain the dispersion relation for electromagnetic waves:

$$\omega^2 = k_z^2 c^2 + \omega_{pe}^2. \tag{1.75}$$

### 1.4.4 Linearized Vlasov Waves in Magnetized Plasma

We begin with a derivation of the general dispersion relation for waves in an infinite, uniform, and magnetized plasma on the basis of eqs. 1.58 and 1.64, where we have assumed a uniform external magnetic field $\boldsymbol{B}_0 = B_0\boldsymbol{e}_z$, and the equilibrium distribution function is $f_{\alpha 0}(\boldsymbol{v}) = f_{\alpha 0}(v_{\perp,v_z})$ (the plasma is isotropic in the x-y dimensions). Note that the situation is 3-dimensional anisotropic. Thus, we expect the conductivity to be in the form of a tensor: $\boldsymbol{J}_{1k} = \overleftrightarrow{\sigma} \cdot \boldsymbol{E}_{1k}$. We write the general form as:

$$\boldsymbol{J}_{1k} = \overleftrightarrow{\sigma} \cdot \boldsymbol{E}_{1k} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} E_{1kx} \\ E_{1ky} \\ E_{1kz} \end{pmatrix}. \tag{1.76}$$

Without loss of generality (for a plasma isotropic in x, y), we let $\boldsymbol{k} = k_\perp \boldsymbol{e}_x + k_z \boldsymbol{e}_z$. Then from the field equation:

$$\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_{1k}) + \frac{\omega^2}{c^2}\boldsymbol{E}_{1k} = -\frac{4\pi i\omega}{c^2}\boldsymbol{J}_{1k},$$

we have the following equations by the x, y, z components:

$$\begin{aligned} &E_{1kx}(1 - \frac{k_z^2c^2}{\omega^2} + \frac{4\pi i}{\omega}\sigma_{xx}) + E_{1ky}\frac{4\pi i}{\omega}\sigma_{xy} + E_{1kz}(\frac{k_\perp k_z c^2}{\omega^2} + \frac{4\pi i}{\omega}\sigma_{xz}) = 0, \\ &E_{1kx}\frac{4\pi i}{\omega}\sigma_{yx} + (1 - \frac{k^2c^2}{\omega^2} + \frac{4\pi i}{\omega}\sigma_{yy})E_{1ky} + \frac{4\pi i}{\omega}\sigma_{yz}E_{1kz} = 0 \\ &E_{1kx}(\frac{k_\perp k_z c^2}{\omega^2}) + E_{1ky}\frac{4\pi i}{\omega}\sigma_{zy} + E_{1kz}(1 - \frac{c^2k_\perp^2}{\omega^2} + \frac{4\pi i}{\omega}\sigma_{zz}) = 0. \end{aligned} \tag{1.77}$$

Eq. 1.77 can be written in a matrix form:

$$\overleftrightarrow{D} \cdot \boldsymbol{E}_{1k} = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix} \begin{pmatrix} E_{1kx} \\ E_{1ky} \\ E_{1kz} \end{pmatrix} = 0, \tag{1.78}$$

i.e.,

$$\begin{pmatrix} 1 - \frac{k_z^2c^2}{\omega^2} + \frac{4\pi i}{\omega}\sigma_{xx} & \frac{4\pi i}{\omega}\sigma_{xy} & \frac{k_\perp k_z c^2}{\omega^2} + \frac{4\pi i}{\omega}\sigma_{xz} \\ \frac{4\pi i}{\omega}\sigma_{yx} & 1 - \frac{k^2c^2}{\omega^2} + \frac{4\pi i}{\omega}\sigma_{yy} & \frac{4\pi i}{\omega}\sigma_{yz} \\ \frac{k_\perp k_z c^2}{\omega^2} & \frac{4\pi i}{\omega}\sigma_{zy} & 1 - \frac{c^2k_\perp^2}{\omega^2} + \frac{4\pi i}{\omega}\sigma_{zz} \end{pmatrix} \begin{pmatrix} E_{1kx} \\ E_{1ky} \\ E_{1kz} \end{pmatrix} = 0. \tag{1.79}$$

For the above equation to be solvable, the determinant of $\overleftrightarrow{B}$ mush be 0:

$$det(\overleftrightarrow{D}) = \begin{pmatrix} D_x x & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix} = 0. \tag{1.80}$$

The above equation is the most comprehensive form of the dispersion relation.

Though we have obtained the general form of the dispersion relation, the conductivity tensor $\overleftrightarrow{\sigma}$ is still unknown. To obtain the specific expression, we need to work on the equations for particle dynamcis.

Define $\tau = t - t'$, we have the $f_{\alpha 1k}$ expression in terms of $\tau$:

$$f_{\alpha 1k} = -\frac{q_\alpha}{m_\alpha}\int_{-\infty}^{0} d\tau[\boldsymbol{E}_{1k} + \frac{1}{c}\boldsymbol{v}'(\tau) \times \boldsymbol{B}_{1k}] \cdot \nabla_{\boldsymbol{v}'} f_{\alpha 0}(\boldsymbol{v}') \cdot e^{i\omega\tau + i\boldsymbol{k}\cdot[\boldsymbol{x}'(\tau) - \boldsymbol{x}]}. \tag{1.81}$$

Using $\boldsymbol{k} = k_\perp \boldsymbol{e}_x + k_z \boldsymbol{e}_z$ and the geometry equations in eq 1.53, we have:

$$\begin{aligned}
\boldsymbol{k} \cdot [\boldsymbol{x}'(\tau) - \boldsymbol{x}] &= k_\perp [x'(\tau) - x] + k_z [z(\tau) - z] \\
&= \frac{k_\perp v_\perp}{\Omega_\alpha} [sin(\phi - \Omega_\alpha \tau) - sin\phi] + k_z v_z \tau \\
\Rightarrow e^{i\omega\tau + \boldsymbol{k}[\boldsymbol{x}'(\tau) - \boldsymbol{x}]} &= e^{-i(\omega - k_z v_z) - i\frac{k_\perp v_\perp}{\Omega_\alpha}[sin(\phi - \Omega_\alpha \tau) - sin\phi]}.
\end{aligned}$$

Using the Bessel function:

$$e^{\pm ixsin\theta} = \sum_{s=-\infty}^{\infty} J_s(x) e^{\pm is\theta},$$

we have:

$$\begin{aligned}
e^{-i\frac{k_\perp v_\perp}{\Omega_\alpha} sin(\phi - \Omega_\alpha \tau)} &= \sum_{s=-\infty}^{+\infty} J_s(\frac{k_\perp v_\perp}{\Omega_\alpha}) \cdot e^{-is(\phi - \Omega_\alpha \tau)} \\
e^{i\frac{k_\perp v_\perp}{\Omega_\alpha} sin\phi} &= \sum_{s'=-\infty}^{+\infty} J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha}) e^{is'\phi}.
\end{aligned}$$

Thus, we may write:

$$\begin{aligned}
e^{-i\omega + i\boldsymbol{k}\cdot[\boldsymbol{x}'(\tau) - \boldsymbol{x}]} &= e^{-i(\omega - k_z v_z)\tau} \cdot e^{-i\frac{k_\perp v_\perp}{\Omega_\alpha}[sin(\phi - \Omega_\alpha \tau) - sin\phi]} \\
&= \sum_s \sum_{s'} J_s(\frac{k_\perp v_\perp}{\Omega_\alpha}) J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha}) e^{-i(\omega - k_z v_z)\tau - is(\phi - \Omega_\alpha) + is'\phi} \\
&= \sum_s \sum_{s'} J_s(\frac{k_\perp v_\perp}{\Omega_\alpha}) J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha}) e^{-i(\omega - k_z v_z - s\Omega_\alpha)\tau + i(s' - s)\phi}.
\end{aligned} \tag{1.82}$$

Since $f_{\alpha_0}(\boldsymbol{v}') = f_{\alpha 0}(v'_\perp, v'_z)$ and $v'_\perp (= v_\perp)$ and $v'_z (= v_z)$ are constants of motion, we have:

$$\begin{aligned}
\nabla_{\boldsymbol{v}'} f_{\alpha 0}(\boldsymbol{v}') &= \nabla_{\boldsymbol{v}} f_{\alpha 0}(\boldsymbol{v}) \\
&= \frac{\partial f_{\alpha 0}}{\partial v_\perp} \boldsymbol{v}_\perp + \frac{\partial f_{\alpha 0}}{\partial v_z} \boldsymbol{v}_z \\
&= 2\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} \boldsymbol{v}_\perp + 2\frac{\partial f_{\alpha 0}}{\partial v_z^2} \boldsymbol{v}_z
\end{aligned} \tag{1.83}$$

Then, we have:

$$\begin{aligned}
\boldsymbol{E}_{qk} \cdot \nabla_{\boldsymbol{v}'} f_{\alpha 0}(\boldsymbol{v}') &= 2(E_{1kx}\boldsymbol{e}_x + E_{1ky}\boldsymbol{e}_y + E_{1kz}\boldsymbol{e}_z) \cdot (\frac{\partial f_{\alpha 0}}{\partial v_\perp^2} \boldsymbol{v}_\perp + \frac{\partial f_{\alpha 0}}{\partial v_z^2} \boldsymbol{v}_z) \\
&= 2(E_{1kx} v_x + E_{1ky} v_y) \frac{\partial f_{\alpha 0}}{\partial v_\perp^2} + 2E_{1kz} v_z \frac{\partial f_{\alpha 0}}{\partial v_z^2}.
\end{aligned} \tag{1.84}$$

With Faraday's law, we have $\boldsymbol{B}_{1k} = \frac{c}{\omega} \boldsymbol{k} \times \boldsymbol{E}_{1k}$. Then

$$\boldsymbol{v} \times \boldsymbol{B}_{1k} = \frac{c}{\omega} \boldsymbol{v} \times (\boldsymbol{k} \times \boldsymbol{E}_{1k}) = \frac{c}{\omega} [\boldsymbol{k}(\boldsymbol{v} \cdot \boldsymbol{E}_{1k}) - (\boldsymbol{v} \cdot \boldsymbol{k}) \boldsymbol{E}_{1k}]. \tag{1.85}$$

Combine eqs 1.83 and 1.85, we have:

$$
\begin{aligned}
&\frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B}_{1k})\cdot\nabla_{\boldsymbol{v}'}f_{\alpha 0}=\frac{2}{\omega}[\boldsymbol{k}(\boldsymbol{v}\cdot\boldsymbol{E}_{1k})-(\boldsymbol{v}\cdot\boldsymbol{k})\boldsymbol{E}_{1k}](\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}\boldsymbol{v}_\perp+\boldsymbol{v}_z\frac{\partial f_{\alpha 0}}{\partial v_z^2})\\
&=\frac{2}{\omega}\{\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}[(\boldsymbol{v}\cdot\boldsymbol{E}_{1k})(\boldsymbol{k}\cdot\boldsymbol{v}_\perp)]+\frac{\partial f_{\alpha 0}}{\partial v_z^2}[(\boldsymbol{v}\cdot\boldsymbol{E}_{\boldsymbol{1k}})k_zv_z-(\boldsymbol{k}\cdot\boldsymbol{v})v_zE_{1k}]\}\\
&=\frac{2}{\omega}\{\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}[(v_xE_{1kx}+v_yE_{1ky}+v_zE_{1kz})k_\perp v_x-(k_\perp v_x+k_zv_z)(E_{1kx}v_x+E_{1ky}v_y)]\\
&\quad+\frac{\partial f_{\alpha 0}}{\partial v_z^2}[(v_xE_{1kx}+v_yE_{1ky}+v_zE_{1kz})k_zv_z-(k_\perp v_x+k_zv_z)v_zE_{1kz}]\}\\
&=\frac{2}{\omega}[\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(-k_zE_{1kx}v_x-k_zE_{1ky}v_y+k_\perp E_{1kz}v_x)v_z+\frac{\partial f_{\alpha 0}}{\partial v_z^2}(v_xE_{1kx}k_z+v_yE_{1ky}k_z-k_\perp v_xE_{1kz})v_z].
\end{aligned}
\tag{1.86}
$$

On the other hand, we have (using the eqs 1.53):

$$
\begin{aligned}
&[\boldsymbol{E}_{1k}+\frac{1}{c}(\boldsymbol{v}\times\boldsymbol{B}_{1k})]\nabla_{\boldsymbol{v}'}f_{\alpha 0}=2v_x'X+2v_y'Y+2v_z'Z\\
&=v_\perp[e^{i(\phi-\Omega_\alpha t)}+e^{-i(\phi-\Omega_\alpha t)}]X+v_\perp[e^{i(\phi-\Omega_\alpha t)}-e^{-i(\phi-\Omega_\alpha t)}]Y+2v_zZ
\end{aligned}
\tag{1.87}
$$

Combing the above two equations and compare the $v_x(=v_x')$, $v_y(=v_y')$, and $v_z(=v_z')$ terms, we have:

$$
\begin{aligned}
X&=E_{1kx}\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}+\frac{v_z}{\omega}(k_zE_{1kx}-k_\perp E_{1kz})(\frac{\partial f_{\alpha 0}}{\partial v_z^2}-\frac{\partial f_{\alpha 0}}{\partial v_\perp^2})\\
Y&=E_{1ky}\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}+\frac{v_z}{\omega}k_zE_{1ky}(\frac{\partial f_{\alpha 0}}{\partial v_z^2}-\frac{\partial f_{\alpha 0}}{\partial v_\perp^2})\\
Z&=E_{1kz}\frac{\partial f_{\alpha 0}}{\partial v_z^2}.
\end{aligned}
\tag{1.88}
$$

Then we may write:

$$
\begin{aligned}
&(\boldsymbol{E}_{1k}+\frac{1}{c}\boldsymbol{v}'\times\boldsymbol{B}_{1k})\cdot\nabla_{\boldsymbol{v}'}f_{\alpha 0}e^{-i\omega\tau+i\boldsymbol{k}[\boldsymbol{x}'(\tau)-\boldsymbol{x}]}\\
&=\{v_\perp[e^{i(\phi-\Omega_\alpha t)}+e^{-i(\phi-\Omega_\alpha\tau)}]X+v_\perp[e^{i(\phi-\Omega_\alpha\tau)}-e^{-i(\phi-\Omega_\alpha\tau)}]Y+2v_zZ\}\cdot\\
&\sum_s\sum_{s'}J_s(\frac{k_\perp v_\perp}{\Omega_\alpha})J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha})e^{-i(\omega-k_zv_z-s\Omega_\alpha)\tau+i(s'-s)\phi}\\
&=v_\perp X[e^{i(\phi-\Omega_\alpha t)}+e^{-i(\phi-\Omega_\alpha\tau)}]\cdot\sum_s\sum_{s'}J_s(\frac{k_\perp v_\perp}{\Omega_\alpha})J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha})e^{-i(\omega-k_zv_z-s\Omega_\alpha)\tau+i(s'-s)\phi}\\
&+v_\perp Y[e^{i(\phi-\Omega_\alpha t)}-e^{-i(\phi-\Omega_\alpha\tau)}]\cdot\sum_s\sum_{s'}J_s(\frac{k_\perp v_\perp}{\Omega_\alpha})J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha})e^{-i(\omega-k_zv_z-s\Omega_\alpha)\tau+i(s'-s)\phi}\\
&+2v_zZ\cdot\sum_s\sum_{s'}J_s(\frac{k_\perp v_\perp}{\Omega_\alpha})J_{s'}(\frac{k_\perp v_\perp}{\Omega_\alpha})e^{-i(\omega-k_zv_z-s\Omega_\alpha)\tau+i(s'-s)\phi}
\end{aligned}
\tag{1.89}
$$

Note that:

$$
\begin{aligned}
&\sum_s \sum_{s'} J_s J_{s'} e^{\pm i(\phi-\Omega_\alpha\tau)} \cdot e^{-i(\omega-k_z v_z - s\Omega_\alpha)\tau + i(s'-s)\phi} \\
&= \sum_s \sum_{s'} J_s J_{s'} e^{-i[\omega-k_z v_z-(s\mp 1)\Omega_\alpha]\tau + i(s'-s\pm 1)\phi} \\
&= \sum_n \sum_{s'} J_{n\pm 1} J_{s'} e^{-i(\omega-k_z v_z - n\Omega_\alpha)\tau + i(s'-n)\phi},
\end{aligned}
\tag{1.90}
$$

we have:

$$
\begin{aligned}
&(\boldsymbol{E}_{1k} + \frac{1}{c}\boldsymbol{v}' \times \boldsymbol{B}_{1k}) \cdot \nabla_{\boldsymbol{v}'} f_{\alpha 0} e^{-i\omega\tau + i\boldsymbol{k}\cdot[\boldsymbol{x}'(\tau)-\boldsymbol{x}]} \\
&= \sum_n \sum_{s'} [v_\perp X(J_{n+1}+J_{n-1}) - iv_\perp(J_{n+1}-J_{n-1}) + 2v_z Z J_n] J_{s'} \underbrace{e^{-i(\omega-k_z v_z - n\Omega_\alpha)\tau}}_{\text{only factor related to } \tau} e^{i(s'-n)\phi}.
\end{aligned}
\tag{1.91}
$$

Then we can write the $f_{\alpha 1k}$ in terms of $\tau$ (eq 1.81) to:

$$
\begin{aligned}
f_{\alpha 1k}(\boldsymbol{v}) &= \frac{q_\alpha}{m_\alpha} \int_{-\infty}^{0} e^{-i(\omega-k_z v_z - n\Omega_\alpha)\tau} d\tau \{[v_\perp X(J_{n+1}+J_{n-1}) - iv_\perp(J_{n+1}-J_{n-1}) + 2v_z Z J_n] J_{s'}\} \\
&\quad e^{i(s'-n)\phi} \\
&= -\frac{q_\alpha}{m_\alpha} \sum_n \sum_{s'} \frac{v_\perp X(J_{n+1}+J_{n-1}) - iv_\perp Y(J_{n+1}-J_{n-1}) + 2v_z Z J_n}{i(\omega - k_z v_z - n\Omega_\alpha)} J_{s'} e^{i(s'-n)\phi}
\end{aligned}
\tag{1.92}
$$

Given the above equation, we can write the current $\boldsymbol{J}_{1k}$ as:

$$
\begin{aligned}
\boldsymbol{J}_{1k} &= \sum_\alpha q_\alpha \int f_{\alpha 1k}(\boldsymbol{v}) \boldsymbol{v} d^3 v \\
&= \sum_\alpha q_\alpha \int_0^{+\infty} v_\perp dv_\perp \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dv_z f_{\alpha 1k}(\boldsymbol{v}) \boldsymbol{v},
\end{aligned}
\tag{1.93}
$$

where $\boldsymbol{v} = v_\perp cos\phi \boldsymbol{e}_x + v_\perp sin\phi \boldsymbol{e}_x + v_z \boldsymbol{e}_z$. There are three components of the current: $J_{1kx}$, $J_{1ky}$, and $J_{1kz}$. We consider the $J_{1kx}$ component first:

$$
J_{1kx} = \sum_\alpha q_\alpha = \int_0^{\infty} v_\perp dv_\perp \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dv_z f_{\alpha 1k}(\boldsymbol{v}) \frac{1}{2} v_\perp (e^{i\phi} + e^{-i\phi}). \tag{1.94}
$$

On the other hand, we have:

$$
J_{1kx} = \sigma_{xx} E_{1kx} + \sigma_{xy} E_{1ky} + \sigma_{xz} E_{1kz}. \tag{1.95}
$$

Therefore, $\sigma_{xx}$ is the sum of the coefficients of all $E_{1kx}$ items combining eq 1.92 and eq 1.94:

$$
\begin{aligned}
\sigma_{xx} &= \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int_0^{\infty} v_\perp d_{v_\perp} \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dv_z \frac{1}{2} v_\perp (e^{i\phi} + e^{-i\phi}) \\
&\cdot \sum_n \sum_{s'} \frac{v_\perp [\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial f_{\alpha 0}}{\partial v_z^2}]}{i(\omega - k_z v_z - n\Omega_\alpha)} (J_{n+1} + J_{n-1}) J_{s'} e^{i(s'-n)\phi}
\end{aligned}
\tag{1.96}
$$

Note that for Bessel functions, $J_{n+1}(\frac{k_\perp v_\perp}{\Omega_\alpha}) + J_{n-1}(\frac{k_\perp v_\perp}{\Omega_\alpha}) = \frac{2n\Omega_\alpha}{k_\perp v_\perp} J_n(\frac{k_\perp v_\perp}{\Omega_\alpha})$, we can then write:

$$\begin{aligned}\sigma_{xx} &= \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int_0^\infty v_\perp d_{v_\perp} \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dv_z \frac{n v_\perp \Omega_\alpha}{k_\perp} \\ &\cdot \sum_n \sum_{s'} \frac{\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial f_{\alpha 0}}{\partial v_z^2}}{i(\omega - k_z v_z - n\Omega_\alpha)} J_n J_{s'} [e^{i(s'-n+1)\phi} + e^{i(s'-n-1)\phi}].\end{aligned} \tag{1.97}$$

We can see that only $s' = n \pm 1$ terms can survive in the $\phi$ integration. Thus, we have:

$$\begin{aligned}\sigma_{xx} &= \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \int_0^\infty v_\perp d_{v_\perp} \int_0^{2\pi} d\phi \int_{-\infty}^{+\infty} dv_z \frac{n v_\perp \Omega_\alpha}{k_\perp} \\ &\cdot \sum_n \sum_{s'} \frac{\frac{\partial f_{\alpha 0}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial f_{\alpha 0}}{\partial v_z^2}}{i(\omega - k_z v_z - n\Omega_\alpha)} J_n \underbrace{(J_{n-1} + J_{n+1})}_{\frac{2n\Omega_\alpha}{k_\perp v_\perp} J_n} \\ &= \sum_\alpha \frac{\omega_{p\alpha}^2}{2} \sum_n \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{2n^2\Omega_\alpha^2}{k_\perp} \frac{\frac{\partial \overline{f_{\alpha 0}}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial f_{\alpha 0}}{dv_z^2}}{i(\omega - k_z v_z - n\Omega_\alpha)} J_n^2,\end{aligned} \tag{1.98}$$

where $\overline{f_{\alpha,0}} = \frac{1}{n} f_{\alpha,0}$, so that $\int \overline{f_{\alpha,0}} d^3v = 1$. Then we can write:

$$\begin{aligned}D_{xx} &= 1 - \frac{k_z^2 c^2}{\omega^2} + \frac{4\pi i}{\omega} \sigma_{xx} \\ &= 1 - \frac{k_z^2 + c^2}{\omega^2} - \frac{2\pi}{\omega} \sum_\alpha \omega_{p\alpha}^2 \sum_n \int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{2n^2\Omega_\alpha^2}{k_\perp} \frac{\frac{\partial \overline{f_{\alpha 0}}}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega} \frac{\partial f_{\alpha 0}}{dv_z^2}}{\omega - k_z v_z - n\Omega_\alpha} J_n^2\end{aligned} \tag{1.99}$$

By similar methods, we can obtain the rest of the components. Results are as follows:

$$
\begin{aligned}
D_{xx} &= 1 - \frac{k_z^2 c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \langle \frac{n^2\Omega_\alpha^2}{k_\perp^2} J_n^2(\frac{k_\perp v_\perp}{\Omega_\alpha})\chi_\alpha\rangle \\
D_{xy} &= -\frac{2\pi i}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \langle \frac{n\Omega_\alpha v_\perp}{k_\perp} J_n(\frac{k_\perp v_\perp}{\Omega_\alpha})\frac{dJ_n(k_\perp v_\perp/\Omega_\alpha)}{d(k_\perp v_\perp/\Omega_\alpha)}\chi_\alpha\rangle \\
D_{xz} &= \frac{k_z k_\perp c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha} \langle \frac{n\Omega_\alpha v_z}{k_\perp} J_n^2(\frac{k_\perp v_\perp}{\Omega_\alpha})\Lambda_\alpha\rangle \\
D_{yx} &= -D_{xy} \\
D_{yy} &= 1 - \frac{(k_\perp^2 + k_\parallel^2)c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \langle v_\perp^2 [\frac{dJ_n(k_\perp v_\perp/\Omega_\alpha)}{d(k_\perp v_\perp/\omega_\alpha)}]^2\chi_\alpha\rangle \\
D_{yz} &= \frac{2\pi i}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \langle v_z v_\perp J_n(\frac{k_\perp v_\perp}{\Omega_\alpha})\frac{dJ_n(k_\perp v_\perp/\Omega_\alpha)}{d(k_\perp v_\perp/\Omega_\alpha)}\Lambda_\alpha\rangle \\
D_{zx} &= \frac{k_\perp k_z c^2}{\omega^2} - \frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \langle v_z \frac{n\Omega_\alpha}{k_\perp} J_n^2(\frac{k_\perp v_\perp}{\Omega_\alpha})\chi_\alpha\rangle \\
D_{zy} &= -\frac{2\pi i}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \langle v_z v_\perp J_n(\frac{k_\perp v_\perp}{\Omega_\alpha})\frac{dJ_n(k_\perp v_\perp/\Omega_\alpha)}{d(k_\perp v_\perp/\Omega_\alpha)}\chi_\alpha\rangle \\
D_{zz} &= 1 - \frac{k_\perp^2 c^2}{\omega^2}\frac{2\pi}{\omega}\sum_\alpha\sum_n \omega_{p\alpha}^2 \langle v_z^2 J_n^2(\frac{k_\perp v_\perp}{\Omega_\alpha})\Lambda_\alpha\rangle,
\end{aligned}
\tag{1.100}
$$

where

$$
\begin{aligned}
\langle F(v_\perp, v_z)\rangle &\equiv \int_0^\infty 2v_\perp dv_\perp \int_{-\infty}^\infty dv_z \frac{F(v_\perp, v_z)}{k_z v_z + \Lambda\Omega_\alpha - \omega} \\
\chi_\alpha &\equiv \frac{\partial \overline{f_{\alpha 0}}(v_\perp, v_z)}{\partial v_\perp^2}(1 - \frac{k_z v_z}{\omega}) + \frac{k_z v_z}{\omega}\frac{\partial \overline{f_{\alpha 0}(v_\perp, v_z)}}{\partial v_z^2} \\
\Lambda_\alpha &\equiv \frac{\partial \overline{f_{\alpha 0}}(v_\perp, v_z)}{\partial v_z^2} - \frac{n\Omega_\alpha}{\omega}[\frac{\partial \overline{f_{\alpha 0}}(v_\perp, v_z)}{\partial v_z^2} - \frac{\overline{f_{\alpha 0}}(v_\perp, v_z)}{\partial v_\perp^2}].
\end{aligned}
\tag{1.101}
$$